Abstract
The paper considers the formation of an average opinion index in a microfounded framework where firms switch between optimism and pessimism with certain transition probabilities. Here, the index represents a general business sentiment or the famous animal spirits, which feed back on themselves but are also influenced by the real interest rate. This concept is combined with three other components: The sentiment determines aggregate investment and thus finally the output gap in the economy; the variations of a so-called inflation climate are introduced into an ordinary Phillips curve; the interest rate is given by a Taylor rule. In this way an alternative model of the new macroeconomic consensus is obtained, which is then reduced to two differential equations. The nonlinearities inherent in the sentiment adjustments can give rise to persistent endogenous cycles as well as multiple equilibria that may be locally stable or unstable. It is also demonstrated how the dynamics are affected by the changes of a financial distress variable.

JEL classification: D 84, E 32, E 37.

Keywords: Business sentiment, herding, inflation climate, endogenous cycles, multiple equilibria.

1. Introduction
It is an unchallenged axiom in almost all of current macroeconomic theory that the decisions of the agents are based on their expectations about the future course of some observable key variables, or rather their value in the next period. If there is any debate at all, it is about whether these expectations are formed in a rational or, as it is called, boundedly rational fashion, or what kind of learning processes might enable the agents to converge to rational expectations.

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It may nevertheless be recalled that this point of view has been seriously called into question by, in particular, Keynes in Chapter 12 of his *General Theory*. He there discusses another elementary “characteristic of human nature,” namely, “that a large proportion of our positive activities depend on spontaneous optimism rather than on a mathematical expectation” (Keynes, 1936, p. 161). Although the chapter is titled “The state of long-term expectations”, Keynes makes it explicit that he is concerned with “the state of psychological expectation” (p. 147). Chapter 12 is actually the place where he speaks of the famous “animal spirits”: “Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as a result of animal spirits—of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities” (p. 161).

Animal spirits are not just whims and moods out of the blue, an imperfection or plain ignorance of human decision makers. In the end they are due to the problem that decisions reaching further into the future are not only complex but also fraught with irreducible uncertainty. “About these matters”, Keynes writes at another place to clarify the basic issues of the *General Theory*, “there is no scientific basis on which to form any calculable probability whatever” (Keynes, 1937, p. 114). To cope with this fundamental fact when inevitably a decision has to be taken, and “to behave in a manner which saves our faces as rational economic men” (*ibid.*), Keynes makes reference to “a variety of techniques”, or “principles”. From the three central points that he notes we may here quote the last one, which in today’s language is furthermore reminiscent of herding behaviour:

“Knowing that our own individual judgment is worthless, we endeavor to fall back on the judgment of the rest of the world which is perhaps better informed. That is, we endeavor to conform with the behavior of the majority or the average. The psychology of a society of individuals each of whom is endeavoring to copy the others leads to what we may strictly term a *conventional* judgment.” (Keynes, 1937, p. 114; his emphasis).

Other and scientifically soberer terms than animal spirits, which also allude to this ele-

1 As Farmer (2008) remarks, the idea that business cycles might be driven in part by waves of optimism or pessimism was not new to Keynes and can be traced at least as far back as Henry Thornton (1802, p. 75), unless David Hume (1739, Part iv, Section vii). The rest of Farmer’s survey is, however, concerned with the incorporation of this concept into modern DSGE models, where the term animal spirits is used interchangeably with sunspot equilibria and self-fulfilling prophecies. It will be clear enough that the discussion in this paper has nothing to do with these refinements of rational expectations, where the observations of an exogenous stochastic process induce the agents to coordinate on recurrent switches between multiple equilibria.

2 A more extensive discussion of Keynes’ concepts that can be related to the present paper may be found in Flaschel et al. (1997, Chapter 12.2), or more generally in Minsky (1975, Chapter 3). A good survey of the role of (psychological) expectations and confidence is given in Boyd and Blatt (1988). The recent book by Akerlof and Shiller (2009) on *Animal Spirits* certainly needs no further referencing.
ment of convention, are sentiment (business or consumer sentiment, for example), state of confidence, or climate. Our brief sketches may suffice to illustrate that at least in macroeconomic business cycle theory, it should be fruitful to give the same or even a higher priority to an axiom that provides an alternative to the exclusive focus on expectations of specific macroeconomic variables: the long-term decisions of the agents are based on sentiment.

Even if one may be sympathetic to this intention in its generality, the discussion so far may appear rather vague and it is quite another matter how some crucial aspects of it should be translated into the language of rigorous formal modelling. In fact, the macroeconomic literature in this direction is rather limited, especially if it is taken into account that the abovementioned majority of, say, optimistic agents implies the existence of a minority of more pessimistic agents. In other words, if it is taken into account that a more ambitious modelling should allow for heterogeneity among the agents, in that they may entertain different individual attitudes, act on them, and eventually switch between them.

A priori, a huge variety of creative approaches is conceivable to capture some of these ideas. This principal openness is at the same time a great problem, since it involves the risk of scattered research, incomparable specifications, or just arbitrariness. After all, it is a major advantage of the rational expectations paradigm that it constitutes a unified framework upon which most of contemporary macroeconomics have agreed to work in. Therefore, it is the aim of the present paper to take a step in developing a ‘canonical’ (Lux, 2009, p. 640), though highly stylized framework that allows us to formalize the basic idea of the sentiment axiom with its behavioural heterogeneity. Moreover, the composition of the population of agents should endogenously vary over time, where one—but not the only—cause for waves of optimism and pessimism is a herding mechanism. We could thus also say that we strive for an explicit microfoundation for the dynamic adjustments of a macroscopic sentiment variable. As it is done, it will also be seen that this approach is flexible and can easily be integrated into existing macrodynamic models.

Useful groundwork for such an endeavour has begun a quarter of a century ago in a stimulating book in the social sciences by Weidlich and Haag (1983). Unfortunately, their concepts have not found their way into post-Keynesian or other variants of heterodox macroeconomic theory. One reason for this, in short, is that this research was originally presented in very technical ways, another that the applications dealt with somewhat detached or “exotic” macroeconomic themes or took place outside macroeconomics, in small-scale asset pricing models. The most important reference regarding the latter is Lux (1995, 1998), and indeed the formal structure of the model to be put forward here shows close parallels to these financial market models.

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3 For a more detailed discussion of these points, see Franke (2008a, pp. 239f).
The fundamental idea of the Weidlich–Haag–Lux approach is that the agents choose between two attitudes in a probabilistic manner. For our present purpose, we are concerned with an optimistic and pessimistic investment attitude of firms. The transition probabilities for these switches are not fixed but depend on the current state of the economy. Specifically, they change with the composition of the population of firms, which can give rise to a destabilizing herding dynamics, and with the real rate of interest, which incorporates the chief stabilizing device of current theorizing.4 Formulating these concepts in continuous time and letting the number of firms tend to infinity, a deterministic differential equation for a macroscopic business sentiment variable is obtained. In conjunction with the rest of model, the equation exhibits a, we may say, natural nonlinearity that can produce fairly different dynamic scenarios. In the main, these are global stability of a unique steady state position, persistent cyclical behaviour around this steady state, and two distinct locally stable equilibria with high and low economic activity, respectively, while the steady state is unstable.

On the whole, our model will contain the constituent ingredients of the so-called new macroeconomic consensus, without employing its common rational expectations. Thus, it includes (1) an equation determining the output gap, which with some simplifying assumptions will directly follow from the adjustment equation for the business sentiment. (2) A Phillips curve, where the inflationary expectations are replaced with the notion of a general inflation climate and the supposition of a second adjustment equation for it. And (3) an ordinary Taylor rule of the central bank to set the nominal rate of interest. With the words from above, we can then say that this model introduces a microfounded business sentiment, or animal spirits for that matter, into the framework of the new macroeconomic consensus. Despite its rich interpretational potential, the dynamics can be most conveniently described by a planar system of two differential equations.5

4 In the conception of the “animal spirits” in de Grauwe (2010), the agents likewise choose between a pessimistic and optimistic attitude. However, in its intention to be more directly comparable to the (hybrid) New-Keynesian standard model, his approach differs in the following details. (1) The switching process is specified as a discrete choice mechanism along the lines of Brock and Hommes (1997); (2) optimism and pessimism relate to the predictions of next month’s output gap and inflation rate; (3) the changes in the attitudes are based on their mean square prediction errors, so that, in particular, there is no (direct) herding. Despite these differences and although de Grauwe does not discuss the notion of a sentiment proper, at a more general level we would nevertheless say that his and the present modelling approach share a similar spirit.

5 A forerunner of the present model is Franke (2008a), where the sentiment adjustment equation is integrated into a Goodwinian framework and so yields an enhanced version of persistent income distribution growth cycles. Several other macro models that can be interpreted as being sentiment-driven have recently been designed by Frank Westerhoff (e.g., Westerhoff, 2008, 2010; Westerhoff and Hohnisch, 2007, 2010; Hohnisch and Westerhoff, 2008). We may, however, claim that our present specifications are (perhaps) more elegant, admit a full analytical treatment, and in particular seem better suited to provide a standardized approach to modelling a sentiment dynamics.
The remainder of the paper is organized as follows. The next section details the concept of the transition probabilities for the agents’ switching between optimism and pessimism, and subsequently the adjustment equation for the macroscopic sentiment variable to which they give rise. Section 3 integrates the latter into a macroeconomic framework. Innovative elements are here a so-called adaptive inflation climate entering the Phillips curve, and the credibility of the central bank that in addition to the strength of the herding mechanism will also play an important role for the dynamic properties of the model. Section 4 collects some preparatory analytical results. Section 5 focusses on the conditions for persistent cyclical behaviour, while Section 6 describes the functioning of these fluctuations. Multiple equilibria and multiple attractors are investigated in Section 7. Section 8 discusses how the two-dimensional dynamics may be affected by variations of an exogenous shift parameter that may represent financial distress. These scenarios may serve as prelude to a later augmentation of the model by a financial sector. Section 9 concludes, and an appendix contains the mathematical proofs of the formal propositions in the text.

2. The concept of the business sentiment and its adjustments

Let the business sector consist of \(2N\) firms, whose number remains fixed over time. In each point in time a single firm can be characterized as being either optimistic or pessimistic about the future prospects of the economy. Designating an optimistic and pessimistic attitude by \((+)\) and \((-)\), respectively, let \(n_t^+, n_t^-\) be the number of optimistic and pessimistic firms at time \(t\) \((n_t^+ + n_t^- = 2N)\). Next, put \(n_t = (n_t^+ - n_t^-)/2\) and define \(x_t = n_t/N\). Supposing that there is no systematic relationship between attitudes and the size of the firms, the index \(x_t\) represents the average attitude of the firms. Generally, it may be referred to as the ‘animal spirits’ in the business sector or, more seriously, the business sentiment. As detailed below, this variable will determine investment and thus the level of economic activity.

In the light of some earlier presentations in the literature (e.g. Weidlich and Haag, 1983, or Lux, 1995), it may be explicitly confirmed that \(x_t\) is the index actually prevailing at time \(t\) (and not some mean value in a stochastic system). Obviously, \(-1 \leq x_t \leq 1\); optimism and pessimism balance in a state \(x_t = 0\); and at \(x_t > 0\) \((x_t < 0)\) optimistic \((\text{pessimistic})\) firms form a majority.

Firms may change their attitude over time. To begin with, let us consider this process in discrete time with adjustment periods of length \(\Delta t > 0\). The changes will depend on a great variety of macroscopic aspects and idiosyncratic circumstances, which one will not want to specify in all of their details. It rather seems suitable to introduce random elements in this respect, in order to keep the modelling simple and to avoid arbitrary assumptions. Therefore, the basic concept to describe the changes in the business sentiment are the transition probabilities of the individual firms: at time \(t\), let \(p_t^+\) be the
probability per unit of time that a firm changes from pessimistic to optimistic, and $p_t^{+−}$ the probability for an opposite change. More exactly, $\Delta t p_t^{+−}$ is the probability that a firm that is pessimistic at $t$ will have become optimistic at the end of the period at $t+\Delta t$, and vice versa $\Delta t p_t^{−+}$ for an optimistic firm. These probabilities are assumed to be uniform across the population.\(^6\) They are, however, not fixed but will vary themselves in response to the evolution of certain other macro variables in the economy.

The populations shares of the optimistic and pessimistic firms are related to the index $x_t$ by, respectively, $n_t^+ / 2N = (1 + x_t) / 2$ and $n_t^- / 2N = (1 − x_t) / 2$.\(^7\) To derive the law governing the adjustments of $x_t$ we assume a sufficiently large population, so that the intrinsic noise from different realizations when the individual agents apply their random mechanism can be neglected. In this way, the changes in the two groups are directly given by their size multiplied by the transition probabilities. Accordingly, the population share of the optimists decreases by $\Delta t p_t^{+−} (1 + x_t) / 2$ due to the firms leaving this group, and it increases by $\Delta t p_t^{−+} (1−x_t) / 2$ because of the pessimists who newly join it. With signs reversed, the same holds true for the population share of the pessimistic firms.\(^8\) The net effect on the majority index gives rise to a deterministic adjustment equation, i.e., $x_{t+\Delta t} = x_t + \Delta t [(1−x_t) p_t^{−+} − (1+x_t) p_t^{+−}]$. As it will be more convenient for us to work in continuous time, we let the adjustment period $\Delta t$ shrink to zero and obtain a differential equation for the business sentiment:

$$\dot{x} = (1−x) p^{−+} − (1+x) p^{+−} \quad (1)$$

The effects determining the evolution of the two transition probabilities in (1) are summarized in a switching index $s$. An increase of $s$ is supposed to increase the probability that a pessimist becomes optimistic, and to decrease the probability that an optimist becomes a pessimist. Assuming that the relative changes of $p^{−+}$ and $p^{+−}$ in response to the changes in $s$ are linear and symmetrical, the specification of the transition probabilities reads,\(^9\)

\(^6\) With more information about a firm, its behaviour could be described in terms of conditional probabilities, or each firm could be equipped with a transition probability of its own. Intuition says that these firm-specific probabilities would average out across the whole population, and that the outcome could indeed be treated as a function of certain macroeconomic variables (in the way specified below). Under some mild conditions, it should be possible to employ the Central Limit Theorem and prove this mathematically. Simone Alfarano made some exercises in a related framework to spell out the details (private communication).

\(^7\) To see this, define $n_t = (n_t^+ − n_t^-) / 2 = x_t N$, write the identity $n_t^+ + n_t^- = 2N$ as $n_t^+ / 2 = N - n_t^- / 2$ and add $n_t^+ / 2$ on both sides of this equation. This yields $n_t^+ = N + n_t$ and, after division by $2N$, the first relationship. The second follows analogously.

\(^8\) In contrast to the more elaborate treatment in Lux (1995, 1997), this reasoning, which can also be found in Lux (1998, p. 149), is sufficient for an infinite population. A rigorous mathematical argument, which begins with a finite population size and the intrinsic noise it implies, is spelled out in Franke (2008a or 2008b).

\(^9\) The precise hypothesis is $dp^{−+} / p^{−+} = \alpha ds$ and $dp^{+−} / p^{+−} = −\alpha ds$ for some constant $\alpha$. 

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\[ p^{+} = p^{+}(s) = \nu \exp(s), \quad p^{-} = p^{-}(s) = \nu \exp(-s) \quad (2) \]

where ‘exp’ is the exponential function and \( \nu \) a positive constant. A special feature of (2) is \( p^{+} = p^{-} = \nu > 0 \) when, hypothetically, \( s = 0 \). Hence even in the absence of active feedback forces in the switching index, or when the different feedback variables behind \( s \) neutralize each other, the individual agents will still change their strategy with a positive probability. These reversals, which can occur in either direction, are ascribed to idiosyncratic circumstances. Although they appear as purely random from a macroscopic point of view, in the aggregate they will only cancel out in a balanced state when \( x = 0 \).

In this type of partial adjustments, the state \( x = 0 \) is moreover globally attractive, since \( \dot{x} = -2\nu x \). At the heart of the transition probability mechanism, we have therefore identified a mean-reverting tendency.

Accordingly, possible instabilities of a balanced state of optimism and pessimism will be due to the dynamics governing the switching index \( s \). For these time-varying values, the coefficient \( \nu \) measures the general responsiveness of the transition probabilities to the arrival of new information, as it is summarized by \( s \). So \( \nu \) may be characterized as a flexibility parameter (Weidlich and Haag, 1983, p. 41).

Substituting (2) in (1), the sentiment dynamics (SD) is more compactly described by

\[ \dot{x} = \nu [ (1-x) \exp(s) - (1+x) \exp(-s) ] \quad (SD) \]

We can make an immediate observation on this equation, which has an important implication for the global behaviour of the business sentiment. If it is assumed, or derived from a more specific analysis of a full dynamic model, that the switching index remains bounded over time, then the boundaries \( \pm 1 \) are repelling. For example, in a situation where more and more firms become optimistic, the absolute number of their switches to pessimism, even if the individual transition probabilities are low (but still bounded away from zero), will eventually exceed the number of the remaining pessimists that may switch to optimism. More technically, the first term in square brackets in (SD) tends to vanish as \( x \to 1 \) while the second term is positively bounded away from zero, so that finally the derivative of \( x \) will turn negative and the number of optimists begins to diminish. The argument for \( x \to -1 \) is analogous.

It can thus be concluded that, from a global point of view, the sentiment dynamics has built in a self-stabilization, which applies quite independently of the specific modelling context. We also wish to emphasize that this mechanism is based on a reasoning that refers to the necessarily changing diversity of firms at the micro level. Apart from the concept of individual transition probabilities and the exponential function in their which may be unity without loss of generality (since \( s \) could be arbitrarily scaled). Integrating these relationships with, technically speaking, an integration constant \( \nu \) yields (2). Note that we need not bother about the size of the expressions in (2). It is \( \Delta t \ p^{+}(s) \) and \( \Delta t \ p^{-}(s) \) that must not exceed unity, which in the discrete-time version of (1) is satisfied if only \( \Delta t \) is small enough (and \( s \) is bounded, as it turns out to be).
specification (2), which at that place amounted to a natural linearity assumption with respect to their relative rather than absolute changes, no further extrinsic nonlinearity is required for this global stabilization mechanism to work out.

For the model to be set up in this paper, the determination of the switching index entering (SD) distinguishes two concepts. The first is the concept of self-reference, which grasps the idea that waves of optimistic and pessimistic sentiment are generated by means of a self-exciting mechanism. Accordingly, the probability of switching from pessimistic to optimistic is larger than in the opposite direction if the prevailing opinion in the population of firms is already optimistic (and vice versa). Constructing the index $s$ in an additive way, the first term in $s$ is therefore given by the business sentiment $x$, multiplied by a constant coefficient $\phi_x$ that measures how strongly firms tend to join the current majority. Borrowing the expression from the financial markets literature, $\phi_x$ can be referred to as the model’s herding parameter.\footnote{The firms might also have a general predisposition towards optimism or pessimism, which could be easily captured by including a positive or negative constant in the feedback index $s$. If desired, this aspect might be covered by the distress parameter $d$ below.}

While the first concept essentially constitutes a process of contagion or herding, as it has just been called, we additionally consider a hetero-referential mechanism. It takes account of the more “objective” factors that will induce firms to change their attitude. These factors may reinforce or put a curb on the herding dynamics, or they may even reverse it.\footnote{Equation (SD) together with a suitable specification of the switching index taking self- and hetero-reference into account can also be estimated on empirical sentiment indices obtained from survey data. This has been successfully done by Franke (2008b) using nonlinear least squares for a discrete-time version of (SD), and by Lux (2009) or Ghongadze and Lux (2009) who in a more elaborate approach compute a transient density function which can then serve as an input to a likelihood function.} Here we assume that firms pay attention to three macroeconomic variables. First, firms tend to become more optimistic as economic activity increases. Representing the latter by the output gap $y$, the percentage deviation of total output from potential (or ‘normal’) output, and measuring its impact by a nonnegative coefficient $\phi_y$, the corresponding term in the switching index reads $\phi_y y$. In analogy to the usual reference to economic activity as an argument in a Keynesian investment function, $\phi_y$ may also be called the model’s accelerator parameter.

The second component of hetero-reference can be conceived of as a counterpart of the central stabilizing effect in the so-called new macroeconomic consensus, which is the real interest rate transmission mechanism of monetary policy. We correspondingly suppose that optimism also tends to increase as the real interest rate decreases. Formally, let $i$ be the nominal interest rate, $\pi$ the rate of price inflation, and $r^o$ the long-run equilibrium real rate of interest as it is, by assumption, consistently perceived by both the private sector and the central bank. If the influence of the real interest rate effect is measured by a positive coefficient $\phi_i > 0$, we get the term $-\phi_i (i - \pi - r^o)$ in the switching index. It
may be noted that the present model specifies the transmission of monetary policy in a more indirect way than usual, since the real interest rate does not determine the actual level of the output gap but the rate at which the business sentiment and, consequently, economic activity changes. Given the well-known delays in the effects of monetary policy, this stipulation may not seem unreasonable.\textsuperscript{12}

Third, we want to discuss later how the real economy behaves under different conditions on the financial markets. To this end, we introduce a variable called financial distress $d$, which acts negatively on the switching index and which will be treated as a shift parameter. A zero value of $d$ may indicate circumstances in the financial sector that are largely regarded as ‘normal’, and rising values of $d$ can, in particular, be interpreted as an increasing unwillingness of commercial banks to finance investment, or to refinance positions as the firms’ debt payments become due. Clearly, optimism in the business sector will tend to dwindle, then. Negative values of $d$, on the other hand, could be reflective of a state of increased tranquillity or (expected) profitability in the financial sector.

To sum up, the switching index (SI) is made up of the components,

$$s = \phi_x x + \phi_y y - \phi_i (i - \pi - \pi^o) - d$$

and this expression has to be substituted in (SD).

3. The macroeconomic framework

In this section, the sentiment dynamics is embedded in a macroeconomic framework. Its basic structure corresponds to the new macroeconomic consensus, which requires us to put up three additional building blocks: one determining the output gap, one for the rate of inflation, and one for the nominal rate of interest. Let us begin with the output gap. Viewing total production as essentially demand-driven, we concentrate on fixed investment as the only active component of demand and conceive it as an increasing function of the business sentiment.\textsuperscript{13} Thus, considering the micro level, each individual firm has two investment options, which are given by a low growth rate $g_{\min}$ if the firm is pessimistic, and a high growth rate $g_{\max}$ if it is optimistic. These rates are uniform across the population of firms.

Let $g^o$ be the growth rate of the aggregate capital stock that will prevail in a balanced state of the business sentiment, when $x = 0$. Since the aggregate capital growth rate $g$ equals $g_{\min}$ if $x = -1$ and $g_{\max}$ if $x = +1$, we have $g^o = (g_{\min} + g_{\max})/2$. Generally,

\textsuperscript{12}It would also make sense to have the nominal interest rate entering the switching index. Following the reasoning in, e.g., Fazzari et al. (2008, Section 3.1), this assumes that firms are concerned about their cash flow, where rising interest payments on their outstanding debt decrease the availability of internal financing and so expose the firms to higher risk. We neglect this option because reference to real interest rates is so pervasive in the literature.

\textsuperscript{13}The structure of eq. (3) for the output-capital ratio below would be preserved if capacity utilization were added as another direct determinant of net investment.
recalling the expressions for the population shares from above, a value $x$ of the sentiment gives rise to the aggregate capital growth rate $g = g(x) = g_{max}(1+x)/2 + g_{min}(1-x)/2 = g^o + \beta_{gx} x$, where $\beta_{gx} := (g_{max} - g_{min})/2$.\footnote{Strictly speaking, this simple aggregation rule presumes that optimistic and pessimistic firms have a symmetric size distribution around $g_{min}$ and $g_{max}$, respectively, and that this symmetry is preserved under the ongoing switches between the two groups. Tacitly, similar assumptions are underlying all present small-size models where agents may switch from one type of behaviour to another. Nevertheless, simulations in a consistent analysis would have to specify a large number of firms and keep track of the individual capital stocks in order to check whether the error in the approximation by our short-cut tends to vanish as the population size increases.}

The other components of demand are assumed to be proportional to either the capital stock $K$, which may serve as a trend indicator, or to disposable income, which itself is supposed to be given by total income $Y$ minus a fraction of $K$. We furthermore assume a closed economy and continuous temporary equilibrium on the goods markets, so $Y$ also equals total output.\footnote{All of this was standard in many of the older Keynesian growth models to keep them tractable; see, e.g., the formulation of the (extended) AS–AD textbook model analyzed in Chiarella et al. (2005, Chapter 2), which draws on Sargent (1979, Chapter 5).} With suitable constants $c_1, \ldots, c_4$ and $\sigma$ the private sector’s (constant) propensity to save out of disposable income, the market clearing condition gives $Y = \text{net investment} + \text{capital depreciation} + \text{private consumption} + \text{government spending} = gK + c_1K + (1-\sigma) \cdot \text{disposable income} + c_2K = gK + (c_1+c_2)K + (1-\sigma) (Y - c_3K) = gK + c_4K + (1-\sigma) Y$. Put $\beta_u = c_4/\sigma$ and let $u = Y/K$ denote the output-capital ratio, which indicates the utilization of the capital stock. Dividing the market clearing condition by $K$ and solving it for $u$, a simple multiplier relationship is obtained,

$$u = g(x) / \sigma + \beta_u = (g^o + \beta_{gx} x) / \sigma + \beta_u$$  \hspace{1cm} (3)

Next, define ‘normal’ utilization $u^o$ as the output-capital ratio brought about by the capital growth rate $g^o$ in a balanced state of sentiment, i.e., $u^o := g^o/\sigma + \beta_u$.\footnote{On the basis of the steady state calibration of some macroeconomic key ratios in Chiarella et al. (2005, pp. 85f), we could set $\sigma = 0.069$ and $\beta_u = 0.266$. In this way an annualized steady state growth rate $g^o = 3\%$ yields an equilibrium output-capital ratio $u^o = 0.70$ (for likewise annualized production flows), which it is argued at the same place is a reasonable order of magnitude. However, it should not be concealed that, even if relatively wide variations of $u$ were permitted, the associated deviations of the capital growth rates from $g^o$ be unrealistically low. Or one accepts that a higher savings propensity to mitigate this problem implies unrealistic macroeconomic proportions elsewhere in the background of the model. Problems of this kind are the price one has to pay for the simple specification of the demand side with its fixed proportions, as well as for the convenience of temporary equilibrium on the goods market.} This concept allows us to establish a firm link between utilization and the output gap. As the latter is here reasonably specified as the percentage deviation of total output $Y$ from normal output $Y^o := u^o K$, the output gap can be expressed as $y = (Y - Y^o)/Y^o = (u - u^o)/u^o$. Eq. (3) then leads to $y = (\beta_{gx}/\sigma u^o) x$. Hence, putting $\eta := \beta_{gx}/\sigma u^o$, the temporary IS equilibrium on the goods market conveniently boils down to
\[ y = \eta x \]  
(IS)

which is all we need from now on (i.e., the analysis of the dynamic economy will not refer to the coefficients involved in the derivation of \( \eta \)). It bears emphasizing that the equilibrium condition (IS) as such is rather uninformative. The central forces impacting on economic activity are rather to be sought in the variables that govern the (dynamic) adjustments of the business sentiment in (SD) and (SI). In particular, as already hinted at above, the possibly stabilizing real interest rate effects from a suitable monetary policy are thus of a more indirect nature than in the ordinary models of old- or New-Keynesian origin.

We can thus turn to the second equation of the macroeconomic framework, which determines the rate of price inflation \( \pi \). To this end, we employ a relationship that, with \( \kappa > 0 \) being the ordinary slope coefficient, looks like a normal Phillips curve (PC):

\[ \pi = \pi^e + \kappa y \]  
(PC)

However, the important point to note is that \( \pi^e \) is not meant to represent the expected rate of inflation, where in the common discrete-time framework the expression “expected” refers to the next period, and also in continuous-time formulations the interpretation is more or less explicitly that of a short time horizon. We rather conceive of \( \pi^e \) as a general inflation climate. This term might be directly accepted as an indication that the expectations underlying the price setting process are of a more diffuse nature and are certainly not uniform. Alternatively, we may refer to a more detailed story of boundedly rational and heterogeneous firms in the usual Calvo setting as it has been put forward in Franke (2007, Section 2).  

To breathe life into the notion of the inflation climate, it has to be clarified how \( \pi^e \) is supposed to change over time. To summarize the discussion in Franke (2007, Section 3), \( \pi^e \) is treated as a variable that is predetermined at time \( t \) and changes ‘between periods’ (which are infinitesimally short) as new information arrives. In this part of the model, optimism and pessimism play no essential role. The updating procedure is rather based on the concept of a benchmark rate of inflation towards which the value of \( \pi^e \) is adjusted in a gradual manner. We refer to this principle as the adaptive inflation climate (AIC). In this paper it is assumed that the benchmark is a weighted average of just two rates of inflation: the inflation rate \( \pi \) currently observed and the central bank’s target rate of inflation, \( \pi^* \). Letting \( \gamma \) be the weighting factor for the latter (0 ≤ \( \gamma \) ≤ 1) and \( \alpha > 0 \)

\[ 17 \]  
There, a rate \( \pi^e_f \) of firm \( f \) is interpreted as its expected average of the time-varying rate of inflation over the entire future to come, where the averaging invokes a discounting procedure with probability weights that derive from the expected frequency at which the firm will be able to reset its price. In the end, this is rule-of-thumb guesswork that cannot be based on a consistent mathematical expectation. \( \pi^e \) is then the aggregate of the firms’ heterogeneous estimates.

\[ 18 \]  
First exploratory estimation attempts seem to indicate that this specification is already fully sufficient. Theoretically, the basic idea can be traced back (at least) to Groth (1988, p. 254). It
the speed at which the adjustments take place, the present continuous-time version of AIC reads,

\[ \dot{\pi}^c = \alpha \left[ \gamma \pi^* + (1 - \gamma) \pi - \pi^c \right] \tag{AIC} \]

It is a routine exercise to demonstrate that the value of the inflation climate is determined as a weighted average of the target rate \( \pi^* \) and the entire history of the exponentially weighted inflation rates,

\[ \pi^c = \gamma \pi^* + (1 - \gamma) \alpha \int_{-\infty}^{t} e^{-\alpha(t-s)} \pi(s) \, ds \tag{4} \]

The parameter \( \gamma \) thus indicates the relative importance that the firms attach to the inflation target vis-à-vis past inflation experience. Likewise, we can say that \( \gamma \) expresses the degree to which the inflation climate is anchored on the central bank’s target. In this sense the coefficient, which for simplicity is supposed to be exogenously fixed, can be viewed as measuring the confidence of firms in the monetary policy of the central bank, so that \( \gamma \) may be referred to as the central bank’s credibility.\(^{19}\)

As the third equation of the macroeconomic framework, it remains to specify monetary policy (MP) itself. Here the Taylor rule in its original form (Taylor, 1993) will be good enough for our purpose.\(^{20}\) As the nominal interest rate \( i^o \) corresponding to the equilibrium real rate \( r^o \) should be consistent with the inflation target, i.e. \( r^o = i^o - \pi^* \), this precept can be written as

\[ i = i^o + \mu_\pi (\pi - \pi^*) + \mu_y y, \quad i^o = r^o + \pi^* \tag{MP} \]

Certainly, the two policy coefficients on the inflation and output gap are nonnegative. The condition \( \mu_\pi > 1 \) describes the so-called Taylor principle, which in many New-Keynesian models is (almost) necessary for determinacy of the rational expectations solution, while in some prototype backward-looking models of the new macroeconomic consensus it is necessary for the dynamic (local) stability of the steady state (Chiarella et al., 2005, Chapter 8.3). For this reason, \( \mu_\pi > 1 \) will also be presupposed throughout the present paper. In fact, necessity of the Taylor principle for local stability will carry over to our model, too. On the other hand, as we may anticipate, this condition will by no means prove sufficient for this property.

\(^{19}\) Similar to the sum of the coefficients on the lagged inflation rates in autoregressions or a Phillips curve context, the complementary weight \( (1 - \gamma) \) on past inflation in (4) could be said to measure inflation persistence (conditional on the output gap), when (4) is substituted in (PC).

\(^{20}\) Once in a while, a little tribute may be paid to the ancestors of the basic idea of inflation targeting. After all, it was around 100 years before Taylor that Wicksell built a positive effect of price inflation on the interest rate into his cumulative process determining inflation, and Thornton was saying much the same thing almost another century before Wicksell.
The specification of the Taylor completes the formulation of the model, which is thus constituted by the elements \((SD), (SI), (IS), (PC), (AIC), (MP)\). It is immediately checked that, under normal conditions on the financial markets \((d = 0)\), long-run equilibrium growth at the constant rate \(g^o\) and a zero output gap \(y = 0\) is established by a balanced business sentiment, \(x = 0\), and by inflation being on target, \(\pi = \pi^c = \pi^*\).

As the structural equations have been specified, the model can be reduced to a two-dimensional differential equations system (presupposing that the parameters and, in particular, financial distress \(d\) are fixed). After a bit of straightforward algebraic manipulation, we arrive at the following formulation:

\[
\begin{align*}
\dot{x} &= \nu \{ (1-x) \exp[s(x, \pi^c, d)] - (1+x) \exp[-s(x, \pi^c, d)] \} \\
\dot{\pi}^c &= \alpha [\gamma \pi^* + (1-\gamma)(\pi^c + \kappa \eta x) - \pi^c] \\
s(x, \pi^c, d) &= A x - \phi_i (\mu_{\pi}-1) (\pi^c - \pi^*) - d \\
A &= \phi_x + \phi_y \eta - \phi_i \eta [(\mu_{\pi}-1)\kappa + \mu_y]
\end{align*}
\]

Of course, \(x = 0\) and \(\pi^c = \pi^*\) is also a stationary point of system (5) if \(d = 0\). As will be seen below, depending especially on the composite parameter \(A\), there may exist additional stationary points which, however, will entail a nonzero output gap. Hence they would not constitute a long-run equilibrium proper. Although not formally included in system (5), there would then be forces at work that sooner or later would move the economy out of such a position. This issue will also be part of the discussion later on.

4. Some basic analytical results

As will be seen shortly, system (5) can be highly nonlinear, so that its global behaviour may be fairly different from the local trajectories. An investigation of the global dynamics, which has to complement the results from a local stability analysis, is done by means of phase diagrams. The basic tools for this purpose are the isoclines of the two state variables, that is, the geometric locus of the pairs \((x, \pi^c)\) that give rise to \(\dot{x} = 0\) and \(\dot{\pi}^c = 0\), respectively. Besides, the relative slopes at their point(s) of intersection will also have a bearing on local stability or instability.

For the geometric analysis of the isoclines in the plane, let us draw the business sentiment on the horizontal axis and the inflation climate on the vertical axis. Solving the equation \(\dot{\pi}^c = 0\) in (5) for \(\pi^c\), the isocline of the inflation climate is then immediately computed as

\[
\pi^c = 0 \iff \pi^c = h_{\pi}(x) := \pi^* + (1-\gamma)\kappa \eta x / \gamma
\]

It is thus an upward-sloping straight line through the point \(x = 0\), \(\pi^c = \pi^*\), which tends to become vertical as \(\gamma\) approaches zero. Conversely, rising values of the central bank credibility parameter produce an increasingly flatter slope, such that at perfect credibility.
(γ = 1) the isocline is a horizontal line $π^c = \pi^*$. From the dynamic adjustments of $π^c$ in eq. (5) it is also easily seen that the inflation climate rises at points $(x, π^c)$ below (or to the right of) the isocline, and it decreases in the other half of the phase plane.

For an explicit computation of the $\dot{x} = 0$ isocline, it is useful to work with hyperbolic functions in the sentiment adjustments. The definitions of the hyperbolic sine and cosine (sinh and cosh) allow us to reformulate the first equation in (5) as

$$
\dot{x} = 2\nu \left\{ \frac{\exp(s) - \exp(-s)}{2} - x \frac{\exp(s) + \exp(-s)}{2} \right\} = 2\nu \left[ \sinh(s) - x \cosh(s) \right].
$$

With $\tanh = \sinh / \cosh$ for the hyperbolic tangent, the adjustment equation reads,

$$
\dot{x} = 2\nu \left[ \tanh(s) - x \right] \cosh(s), \quad s = s(x, π^c, d)
$$

By virtue of the positivity of the hyperbolic cosine, the right-hand side of (7) vanishes if $\tanh(s) = \tanh[Ax - \phi_i(\mu_{π} - 1)(π^c - \pi^*) - d] = x$. Applying the inverse function $\text{arctanh}(\cdot)$ to both sides of this equality, using the identity $\text{arctanh}(x) = (1/2) \cdot \ln(1 + x / (1 - x))$, and solving the resulting equation for $π^c$, we obtain:

$$
\dot{x} = 0 \iff π^c = h_x(x, d) := \pi^* + \frac{1}{\phi_i(\mu_{π} - 1)} \left[ Ax - \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right) - d \right]
$$

Since with $x$ fixed, the right-hand side of the differential equation for $x$ in (5) increases if and only if the switching index $s$ increases, and since $s$ is inversely related to $π^c$, we can conclude that in the course of the dynamic process the business sentiment declines at points $(x, π^c)$ above the graph of the function $h_x$, and it rises at points below it.

Recalling the assumption of the Taylor principle, $\mu_{π} > 1$, an increase in financial distress $d$ is thus seen to shift the graph of the isocline downwards in the $(x, π^c)$-plane, without affecting its shape. Given $d$, the expression $h_x(x, d)$ tends to $-∞$ as $x$ approaches 1, and to $+∞$ as the sentiment approaches $-1$ from the right. For a better evaluation of the shape of the isocline, we also need the corresponding partial derivative of the function, which results like

$$
\frac{\partial h_x}{\partial x} = \frac{1}{\phi_i(\mu_{π} - 1)} \left[ A - \frac{1}{1 - x^2} \right]
$$

From this expression it can be inferred that the isocline is everywhere strictly decreasing if the composite parameter $A$ falls short of unity. Otherwise the isocline has a positive slope over an intermediate range of the business sentiment, though it still decreases for $x$ closer to the $±1$ boundaries. Equating the derivative (9) to zero, it is seen that in this case the isocline has exactly one local minimum (maximum) at a negative (positive) value of the sentiment index. Since these features also have an impact on stability and even the number of equilibria, the parameter $A$ will play a crucial role in the local as well as global analysis of system (5).

Before proceeding with the global analysis, let us turn to the local stability properties of system (5), which are determined by the trace and determinant of its Jacobian matrix $J$.  

For their analysis it is convenient to write the two differential equations more compactly as
\[
\dot{x} = F_x(x, \pi^c), \quad \dot{\pi}^c = F_{\pi}(x, \pi^c) \tag{10}
\]
Denoting the partial derivatives as \( F_{xx} = \partial^2 F_x / \partial x \), etc., and using \( \tanh'(s) = 1 / \cosh^2(s) \) for the derivative of the hyperbolic tangent, the Jacobian of (10) is given by
\[
J = \begin{bmatrix}
F_{xx} & F_{x\pi} \\
F_{\pi x} & F_{\pi \pi}
\end{bmatrix}
= \begin{bmatrix}
\frac{2 \nu [A - \cosh^2(s)]}{\cosh^2(s)} & -2 \nu \phi_i (\mu_n - 1) \\
\alpha (1-\gamma) \kappa \eta & -\alpha \gamma
\end{bmatrix} = \begin{bmatrix}
? & - \\
+ & -
\end{bmatrix} \tag{11}
\]
From the sign pattern in (11) one directly obtains a sufficient condition for local stability, which says that in a stationary point \((x, \pi^c)\) the inequality \( A < \cosh^2[s(x, \pi^c, d)] \) is satisfied; this yields a negative sign in the upper-diagonal entry and thus a negative trace and a positive determinant. If \( A < 1 \), these properties hold true in the entire phase plane (since \( \cosh \geq 1 \)), which by virtue of Olech’s theorem even ensure global stability. On the whole, the following statements can be readily derived.\(^{21}\)

**Proposition 1**

With respect to a given level of financial distress \( d \), the following holds for the dynamic system (5).

1. The \( \dot{x} = 0 \) isocline is downward-sloping at a point \((x, \pi^c)\) if and only if the upper-diagonal entry of the Jacobian \( J \) in (11) is negative.

2. If, at a point of rest \((x, \pi^c)\), the \( \dot{x} = 0 \) isocline is downward-sloping, this point is locally asymptotically stable.

3. If \( A < 1 \), system (5) has a unique point of rest, which is globally asymptotically stable.

4. Given that \( 0 < \gamma < 1 \), this point constitutes a long-run equilibrium in which inflation is on target if and only if \( d = 0 \). In this case, the business sentiment is balanced, \( x = 0 \).

It is obvious that sufficiently large values of the composite parameter \( A \) render the trace of \( J \) positive and therefore lead to instability of the steady state. Before turning to this issue, let us consider the main circumstances entailing low values of \( A \). Because of their minor size, we neglect possible variations of the parameter \( \eta \), which is related to an ordinary textbook multiplier, and of the slope coefficient \( \kappa \) in the Phillips curve. Critical are then suitable coefficients in the firms’ switching index and in the monetary policy

\(^{21}\) The proof of this and the other propositions are collected in the appendix.
rule. From the definition of $A$ in (5), the following conditions can be said to be conducive to the system’s global stability:

1. A sufficiently weak herding effect in the adjustments of the business sentiment (parameter $\phi_x$);
2. a sufficiently weak responsiveness of the firms to current economic activity (parameter $\phi_y$);
3. a sufficiently strong responsiveness of the firms to changes in the real interest rate (parameter $\phi_i$);
4. a sufficiently strong inflation targeting on the part of the central bank (policy parameter $\mu_{\pi}$);
5. a sufficiently strong responsiveness of the interest rate to the level of economic activity (parameter $\mu_y$).

If thus $A$ is low enough, the equilibrium may be approached in monotonic or cyclical manner. Over a wide range of parameter values, however, convergence in the latter case is so fast that the oscillatory tendencies are practically negligible. Therefore, for the moment being, we do not go into the finer details of the system’s trajectories in this stability scenario.

5. Unique equilibrium and persistent cycles

This subsection focusses on a unique and unstable equilibrium point that may give rise to persistent cyclical behaviour. Here it has first to be taken into account that instability manifests itself as either a repelling stationary point or a saddle point. As both cases will play an important role in the course of the analysis, it is useful to have a simple geometric criterion that can distinguish between the two. In fact, we only have to consider the slopes of the isoclines at the point where they intersect.

**Proposition 2**

A point of rest is a saddle point if and only if in this point the $\dot{x} = 0$ isocline cuts the upward sloping $\dot{\pi} = 0$ isocline from below.

To organize the discussion of our instability scenarios, we fix financial distress at $d = 0$ in the rest of this and in the next two sections. Under this normality assumption, the stationary point $x = 0$, $\pi_c = \pi^*$ constitutes a long-run equilibrium. Owing to Proposition 1.2, its instability requires that the $\dot{x} = 0$ isocline must not be locally downward sloping. Hence $F_{xx}$ has to be positive in the Jacobian matrix (11), which in turn implies $A > 1$ for the composite parameter $A$ (since $s = s(x, \pi_c, d) = 0$ in the steady state and thus $\cosh(s) = 1$). Furthermore, we want to postpone a discussion of the global implications of a saddle point until later. According to Proposition 2 this is ruled out if, in the
equilibrium, the positive slope of the isocline \( \dot{x} = 0 \) is shallower than that of the straight line on which \( \dot{\pi}^c = 0 \).

The steady state will be repelling if nevertheless the slope of the \( \dot{x} = 0 \) isocline is so steep that the positive entry \( F_{xx} \) in the Jacobian \( J \) exceeds the absolute value of the other diagonal entry \( F_{\pi\pi} \), so that the trace of \( J \) is positive. Otherwise the steady state will continue to be stable, even globally so. Besides the issue of uniqueness, Proposition 3 states the conditions in terms of the model’s parameters under which these two cases may come about.

**Proposition 3**

Suppose \( d = 0, \gamma > 0 \) and put \( B := \phi_i (\mu_\pi - 1) \eta \kappa \). Then, with respect to the composite parameter \( A \) defined in (5), the following holds for this system.

1. The long-run equilibrium \( x = 0, \pi^c = \pi^* \) is the only point of rest if and only if
   \[ 0 < A \leq 1 + \frac{1 - \gamma}{\gamma} B \]

2. The long-run equilibrium is globally asymptotically stable if
   \[ 0 < A < 1 + \min \left\{ \frac{\alpha \gamma}{2\nu}, \frac{1 - \gamma}{\gamma} B \right\} \]

3. The long-run equilibrium is repelling if
   \[ 1 + \frac{\alpha \gamma}{2\nu} < A < 1 + \frac{1 - \gamma}{\gamma} B \]
   where this condition is non-void if \( B = \phi_i (\mu_\pi - 1) \eta \kappa > \frac{\alpha \gamma^2}{2\nu (1 - \gamma)} \).

4. If the condition in the previous point is non-void and, upon a \( ceteris paribus \) increase of \( A \), stability gets lost at \( A = 1 + \alpha \gamma/2\nu \), a Hopf bifurcation occurs.

Regarding the credibility of the central bank, it may be noted in Proposition 3.2 and 3.3 that lower values of \( \gamma \) limit the stability range of the parameter \( A \) (which is independent of \( \gamma \)). Conforming to intuition, this is one aspect in which a loss of confidence in the central bank’s monetary policy may lead to a destabilization.

The expressions \( A \) and \( B \) are not independent of each other since the coefficients entering \( B \) are also contained in \( A \). This aspect, however, can be safely neglected here since in the remainder of the paper the parameters \( \phi_i, \mu_\pi, \eta, \kappa \) in \( B \) will be fixed and only variations in \( \phi_x \) and \( \phi_y \) are discussed. Although the inequalities in the proposition could be easily solved for \( \phi_x + \eta \phi_y \), we prefer the convenience of the slightly more compact formulation.

The last point in the proposition is a first analytical indication of the model’s potential for cyclical behaviour. We nevertheless do not intend to focus on the periodic orbits arising from the Hopf bifurcation, since they are basically a local phenomenon. Let us rather
turn directly to the case of a repelling steady state and study the global dynamics thus generated.

From a remark on the presentation of eq. (5) it is already known that the boundaries \( x = \pm 1 \) are repelling. The discussion of (6) allows us to conclude that, with respect to the \( \dot{\pi}^c = 0 \) isocline in (6), the inflation climate is rising if \( \pi^c \leq h_\pi(-1) \), and falling if \( \pi^c \geq h_\pi(1) \). Hence all trajectories in the phase plane must eventually enter, and then cannot leave, the rectangle given by the four points \((-1, h_\pi(-1)), (1, h_\pi(-1)), (1, h_\pi(1)), (-1, h_\pi(1))\), which means that the conditions of the Poincaré-Bendixson Theorem are satisfied. The fact that additionally the equilibrium is unique and repelling implies sustained cyclical behaviour. This result is summarized in the next proposition.

**Proposition 4**

Suppose \( d = 0, \gamma > 0 \) and that

\[
1 + \frac{\alpha \gamma}{2\nu} < A < 1 + \frac{(1 - \gamma) \phi_i (\mu_\pi - 1) \eta \kappa}{\gamma}
\]

holds true. Then every trajectory starting outside the steady state, which is repelling, converges towards a closed orbit.

Figure 1 illustrates this result. The location of the isoclines and the direction arrows in the resulting four subregions indicate that all motions must be counterclockwise. This implies that not only the inflation climate but also the actual rate of inflation lags behind the output gap; see (PC). The rectangle defined by the (green) dotted lines in the diagram represents the compact invariant set that was the cornerstone of the application of the Poincaré-Bendixson Theorem. While the theorem does not include a statement about the uniqueness of a closed orbit, Figure 1 depicts a situation with a unique limit cycle—the (red) bold line of the closed orbit—which proves to be attractive from both the inside and the outside.

6. The functioning of the cycle

The dynamics sketched in Figure 1 is based on a numerical benchmark parameter scenario. It is reported in Table 1, where it may be noted that the underlying time unit is a year and the rates of interest and inflation as well as the output gap are expressed in percentage terms. The following five parameters were posited straightaway: (a) the flexibility parameter \( \nu \) in the adjustments of the business sentiment, for which a choice of unity means that, in the hypothetical absence of any feedback forces acting on the transition probabilities, a firm would on average change its attitude once a year; (b) the inflation target \( \pi^* \), which is just a matter of scaling; (c) the two policy coefficients \( \mu_\pi \) and \( \mu_y \), for which Taylor’s (1993) original proposal was adopted; and (d) the value
of central bank credibility $\gamma$, which according to the condition in Proposition 3.3 should not be too close to unity if we want to have a unique and repelling steady state.  

The other coefficients were obtained in the course of a trial-and-error procedure by which we sought to calibrate the model to four summary statistics. Making allowances for stronger swings in the turning points that would be likely to occur if random shocks were

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
$\nu$ & Flexibility parameter for the business sentiment 1.00 \\
$\phi_x$ & Herding parameter in the switching index 6.70 \\
$\phi_y$ & Accelerator parameter in the switching index 0.00 \\
$\phi_i$ & Coefficient on the interest rate in the switching index 1.61 \\
$\eta$ & Proportionality factor linking the output gap to $x$ 5.50 \\
$\kappa$ & Slope of the Phillips curve 0.20 \\
$\alpha$ & Adjustment speed for the inflation climate 1.00 \\
$\gamma$ & Credibility of the central bank 0.50 \\
$\mu_\pi$ & Coefficient on inflation in the Taylor rule 1.50 \\
$\mu_y$ & Coefficient on the output gap in the Taylor rule 0.50 \\
$\pi^*$ & Target rate of inflation 2.50 \\
\hline
\end{tabular}
\caption{Numerical benchmark parameter scenario.}
\end{table}

In several exploratory nonlinear least-squares estimations of (PC) and (AIC) with quarterly data for two price deflators over as many as 45 years from 1960 on, we obtained an order of magnitude of $\gamma \approx 0.55$ and $\gamma \approx 0.35$. The same estimates of a rolling sample period of ten years showed a great variability of $\gamma$ between 0.25 and 1.00.
included, the following figures for the present deterministic system appeared reasonable to us: an amplitude of the output gap of $\pm 3.50\%$; an amplitude of the rate of inflation of almost $\pm 1.00\%$; a cycle period of 8.30 years; and a lag of inflation behind the output gap of 0.70 years. The upper two panels in Figure 2, which plot the time series of the two variables on the limit cycle in Figure 1, demonstrate the model’s matching of these “stylized facts”.

The third panel in Figure 2 shows the oscillatory pattern of the switching index $s$ as the driving force in the adjustments of the business sentiment. Including the business sentiment in the diagram would provide no additional information because of its one-to-one correspondence with the output gap (this feature should also be kept in mind in the following discussion). Since by virtue of the Taylor rule (MP) the nominal interest rate has its peaks and troughs between those of $y$ and $\pi$, and by virtue of $\mu_\pi > 1$ the same holds true for the real rate of interest, these series need not be exhibited, either.

![Figure 2: Time series from the limit cycle in Figure 1.](image)

To describe the mechanisms working over the cycle, let us start from the point in the left part of Figure 1 where the closed orbit crosses the $\dot{x} = 0$ isocline. This state constitutes the lower turning of the output gap at $t = 6.45$ in Figure 2. Although the real interest

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23 Other combinations of the parameters may maintain the abovementioned four features but generate motions of $y$ and $\pi$ that are less regular than their sine wave-like pattern in Figure 2.
rate is close to its trough value there, which yields a positive contribution to the switching index \( s = \phi_x x - \phi_i (i - \pi - r^o) \) in equation (SI), the negative herding component is still dominant. Actually, with \( s = -0.74 \), the switching index has just passed its minimum value. Nevertheless, the business sentiment is already beginning to increase. The reason for this is best seen in eq. (7), where \( \tanh(s) \) is negative but less in modulus than the second term \(-x = +0.63 \). Referring to the remark on eq. (2) for the introduction of the switching index, it may thus be said that the recovery of the sentiment is brought about by the basic mean-reverting tendency in the transition probability approach, which in turn could take over since the fall of the switching index has just been stopped by a sufficient lowering of the real interest rate on the part of the central bank.

For a short while after the trough of \( x \), the inflation rate and therefore the real rate of interest continue to decline. Thus, with the beginning increase of the business sentiment, the increase of the switching index gains momentum. In this way, \( \tanh(s) \) in the adjustment equation (7) increases faster than \(-x \) decreases, so that the general sentiment improves further. While the real interest rate soon begins to increase, too, these changes are initially so weak that the increase of \( x \) in the switching index dominates the now negative influence of the changes in \( i - \pi \). Eventually, the sentiment thus becomes positive, at \( t = 8.76 \). From then on, there is a herding toward optimism, which dominates the counteracting effect of the real interest rate in the switching index as well as the mean-reverting tendency back to \( x = 0 \). In other words, the stage of the business cycle before and shortly after the business sentiment \( x \), or the observable output gap \( y \), have crossed the zero line is primarily characterized by a self-reinforcing mechanism where optimism feeds optimism. Given, in particular, the firms’ herding intensity \( \phi_x \) in the composite parameter \( A \) in eq. (5), the policy of the central bank is too weak to prevent this overshooting.

In the course of time, however, the influence of the mean-reverting tendency and the rising real rate of interest become stronger and tend to prevail over the herding feedback. Accordingly, the rise of the business sentiment slows down and finally comes to an end at \( t = 10.58 \) (at the dashed vertical line in Figure 2). It is in this sense that monetary policy is successful. The turning point is then the start of another half-cycle where the motion repeats itself with signs reversed.

It is interesting to study how the features of this limit cycle are affected by a changing credibility of the central bank. Since in this and the following analysis repeated reference is made to the \( \dot{x} = 0 \) and \( \dot{\pi}^c = 0 \) isoclines, which are verbal expressions that might then become somewhat clumsy, let us more compactly denote the two isoclines by \( IC(x) \) and \( IC(\pi^c) \), respectively.

To begin the discussion, consider the reference scenario underlying Figure 1 and suppose a loss of confidence in monetary policy. The decrease in \( \gamma \) induces a counterclockwise rotation of \( IC(\pi^c) \), that is, this isocline has a steeper slope; see eq. (6). The other isocline \( IC(x) \) remains unaltered. In the new scenario, let the economy start from the state \( (x, \pi^c) \)
where the limit cycle in Figure 1 crosses IC($x$) from below at $x > 0$. The crucial point is that with a steeper slope of IC($\pi^c$), the distance of its upper part from this starting point increases. Hence the climate will rise to a higher level until it crosses the IC($\pi^c$) isocline. Furthermore, the distance from there to the left part of the IC($x$) isocline is now shorter than in the reference scenario, which implies that in the downturn the trajectory will cross IC($x$) at a lower value of the business sentiment, and a higher value of the inflation climate. The argument for the second half-cycle is analogous. The periodic orbit to which these motions eventually converge will therefore exhibit a larger amplitude in both $x$ and $\pi^c$, and thus in the output gap $y$ and the rate of inflation $\pi$. It follows that, besides limiting the scope for the steady state’s stability as mentioned in a remark on Proposition 3, a lower credibility has a destabilizing effect in the cyclical global dynamics as well, in a setting where the steady state is repelling anyway.

In addition, it can be argued that the decline in $\gamma$ tends to reduce the period of the cycle. First, the more extreme values of $\pi^c$ in the climate adjustments in (5) increase the speed at which $\pi^c$ changes. Higher values of $\pi^c$ also decrease the switching index $s$ in (5), which therefore reinforces the fall of $x$ in the adjustment equation for the business sentiment; and vice versa for lower values of $\pi^c$ in the recovery of the economy. At least numerically, these effects turn out to be stronger than the possible delays from the longer way the trajectory has to cover in the phase plane. It may thus be said that a lower central bank credibility also increases the volatility of the endogenous cyclical behaviour in the economy.

7. Emergence of multiple equilibria and multiple attractors

We continue to fix financial distress at $d = 0$ in this section and study what happens if the composite parameter $A$ increases such that it finally exceeds $(1-\gamma)B/\gamma$, which according to Proposition 3.3 implies that the long-run equilibrium $x = 0$, $\pi^c = \pi^*$ is no longer the only point of rest. To set the stage, maintain all of the numerical parameters of Table 1 except $\phi_x$. That is, the following changes in $A$ are attributed to *ceteris paribus* variations of the intensity of herding among the firms; recall the definition $A = \phi_x + \phi_y \eta - \phi_i \eta [(\mu_\pi - 1)\kappa + \mu_y]$ to see this.

Our point of departure is the persistent cyclical behaviour under the condition of Proposition 4. The upper-left panel in Figure 3 reproduces the limit cycle of Figure 1 as it was brought about by the benchmark scenario of Table 1 with, in particular, $\phi_x = 6.70$. The other two lines in the panel are IC($\pi^c$) (the straight line) and IC($x$). The former isocline is not affected by the changes in the herding parameter. As can be seen from (9), an increase in $\phi_x$ rather increases the slope of IC($x$) at the steady state. The upper-right panel in Figure 3 illustrates this for $\phi_x = 7.00$, which still satisfies the condition of Proposition 4 (and thus of the uniqueness of the steady state and its repelling character). Again, a unique limit cycle is generated in this way. There is, however, a
certain destabilization of the economy in that the amplitudes of the variables widen. Also the period of this cycle is longer, almost 10 years now.

![Figure 3: Effects of variations of the herding parameter \( \phi_x \).](image)

It is evident that, as the slope of \( \text{IC}(x) \) at the steady state gets steeper with a further increase of \( \phi_x \), this isocline must eventually intersect \( \text{IC}(\pi^c) \) from below. Although not easy to see at the present setting of the axes, this has occurred in the middle-left panel in Figure 3, which is based on \( \phi_x = 7.30 \). To begin with, the panel demonstrates the two immediate consequences of the loss of the uniqueness of the equilibrium. First, the steady state turns into a saddle point, which we know from Proposition 2. Second, since \( \text{IC}(x) \) tends to minus infinity as \( x \to 1 \), and to plus infinity as \( x \to -1 \), there must at least be two additional equilibria. In the diagram, there are also no more than two of them, and Proposition 5 ascertains that this holds true in general.

**Proposition 5**

*Suppose \( d = 0, \gamma > 0 \) and*
Then the steady state \( x = x^0 = 0, \pi^c = \pi^* \) is a saddle point of (5). Furthermore, there are precisely two additional stationary points constituted by \( x^L \) and \( x^H \), where 
\[
|x^L| = x^H \quad \text{and} \quad -1 < x^L < x^0 < x^H < 1
\]
Obviously, \( x^L \) constitutes a low-activity equilibrium with a rate of inflation below the central bank target, \( \pi < \pi^* \), and \( x^H \) a high-activity equilibrium with inflation \( \pi > \pi^* \).

While on account of Proposition 1.2 the outer equilibria would be locally stable if \( \text{IC}(x) \) is downward-sloping there, a positive slope leaves some scope for instability. This is indeed the case for \( \phi_x = 7.30 \) in Figure 3, where \( x^L \) and \( x^H \) are found to be (cyclically) repelling. The (blue) trajectories shown in the diagram are meant to indicate that they will convergence towards a closed orbit, which is again unique by all appearances. The bold (red) trajectory is actually the counterpart of the limit cycle in the second panel with \( \phi_x = 7.00 \)—the emergence of the two additional equilibria had no bearing on this property of the model. The only effect here is a further widening of the amplitudes, and a further noticeable increase of the cycle period to roughly 14 years. If no parameters were to change in the meantime, the long-run dynamics of the economy would therefore be characterized by this lower-frequency cycle.

For completeness it should be added that, despite its uniqueness, not all trajectories starting in disequilibrium will be attracted by the limit cycle. After all, there is still the stable branch of the inner equilibrium saddle point. Of course, this set can have no point in common with the limit cycle. The question where the stable branches emerges has the following somewhat sloppy answer: they have their origin in the two outer equilibria. To describe this one-dimensional manifold more precisely, consider a point \((x, \pi^c)\) on the stable branch with \(0 < x < x^H\). If we move forward in time, the economy converges toward \((0, \pi^* )\). On the other hand, if we hypothetically move backward in time, the system would spiral into \((x^H, \pi^c) = (x^H, h_\pi(x^H))\); and analogously if \(x^L < x < 0\). With a sufficient close-up and vertical stretching, we have checked that practically this geometric locus is very close (though not identical) to the \( \dot{x} = 0 \) isocline between \( x^L \) and \( x^H \).

If the economy starts in a neighbourhood of the stable manifold (or \( \text{IC}(x) \), for that matter), the herding dynamics dominates the inflation climate adjustments. If the start is to the right of the manifold, there is a strong increase in optimism that even overshoots \( x^H \), while pessimism is strongly increasing if the start is slightly to the left of the manifold, with an overshooting of \( x^L \). By the very nature of a stable manifold in a two-dimensional space, the system is here very sensitive to the initial conditions. Nevertheless, these are transitory phenomena. Relatively soon, the long cyclical motion takes over.
A negative slope of IC(x) at the outer equilibria is not necessary for them to become stable. As it turns out, a minor rise of \( \phi_x \) that maintains the positive slope at these points is sufficient for their local stability. Another aspect of these parameter variations is that they, in particular, increase the slope of IC(x) at \( x = 0 \), with the consequence that \( x^L \) and \( x^H \) are shifted to the outside. For a while, however, the outer equilibria will remain inside the previous limit cycle. This means that at these values a closed orbit (somewhat wider than the one generated by \( \phi_x = 7.30 \)) could still exist.

The fourth panel in Figure 3 with the herding parameter \( \phi_x = 7.41 \) illustrates a combination of these features. On the one hand, the increase of \( \phi_x \) is strong enough to render the outer equilibria locally stable. On the other hand, it is moderate enough to preserve the previous stable limit cycle. As a result, we have three local attractors: two equilibrium points and one periodic orbit. The shaded areas in the panel are the basins of attractions of \( (x^L, \pi^L) \) and \( (x^H, \pi^H) \). The trajectories starting in their interior converge to these equilibria in a cyclical manner, while each of the two boundaries is a closed orbit that is repelling on both sides.\(^{24}\)

All trajectories starting outside the shaded areas—and not on a stable branch of the inner saddle point—converge toward the outer closed orbit drawn as the bold (red) line. Hence, to sum up, we have three equilibrium points, two of which are locally attractive, and three periodic motions, one of which is attractive. The coexistence of these equilibrium points or sets, and of the three different attractors, perhaps makes a case like \( \phi_x = 7.41 \) the most attractive scenario. On the other hand, it has to be admitted that the range of \( \phi_x \) that can give rise to this phenomenon is rather limited.

It will be noted that in the diagram for \( \phi_x = 7.41 \), each basin of attraction seems to nestle against the outer limit cycle. The resolution is actually too low to show the remaining space between them. A further increase of \( \phi_x \) enlarges the attraction areas and moves \( x^L \) and \( x^H \) more to the outside as well. Although the outer limit cycle widens, too, the first two effects are stronger. As a consequence, the boundaries of the basins of attraction, i.e. the two unstable limit cycles surrounding them, should eventually touch the outer limit cycle. At that hypothetical moment, however, which because of symmetry is the same for both of the unstable cycles, all three limit cycles dissolve.\(^{25}\)

In fact, the slight increase of the herding parameter from 7.41 to \( \phi_x = 7.42 \) in the lower-left panel of Figure 3 leaves us with the two outer equilibrium points as the only attractors. With the disappearance of all of the closed orbits, their basins of attractions have lost their lenticular shape. For example, the high-activity equilibrium is approached from points far away, whereas other and nearby points diverge from it toward the low-

\(^{24}\) For the rising values of \( \phi_x \), the switch to the stability of the outer equilibria occurs at \( \phi_x \approx 7.373 \). However, the basins of attraction are initially rather small.

\(^{25}\) Additional interesting global phenomena (probably known from the literature) may arise at the bifurcation itself. We leave this issue aside since it is more of a mathematical than economic concern.
activity equilibrium. The two basins of attraction are now adjacent, which means that their common borderline is given by the stable manifold of the interior saddle point. It is not drawn in the diagram since in the range between \( x = \pm 0.75 \) it would be hard to distinguish from the \( \dot{x} = 0 \) isocline.

The role of the stable manifold as a separatrix is more clearly seen in the bottom-right panel, where the herding parameter is moderately increased to \( \phi_x = 7.70 \). The two main differences from the previous panel are that here the outer equilibria are reached in a monotonic rather than cyclical manner, and that the stable manifold, which is the dotted (red) line, has a greater distance from the outer equilibria (so that it can now be better recognized). The expression separatrix emphasizes the fact that in a vicinity of this geometric locus the economy is very sensitive to small shocks in either the sentiment index or the inflation climate, which may deflect a trajectory from its path towards the high-activity equilibrium and turn it around to converge towards the low-activity equilibrium, or vice versa.

To summarize, the gradual \textit{ceteris paribus} variations of the herding parameter from low to high values give rise to a remarkable variety of dynamic scenarios regarding the number of stationary points and periodic orbits as well as the shape of their basins of attractions in case of stability. These features are also important since the weight of herding may not be fixed for overly long periods of time, in particular, if the system tends to settle down in one of the equilibrium points. In addition, we should not neglect \( d \) as the parameter representing the effects originating in the financial sector that have briefly been introduced as “financial distress”. The next section extends the investigation to a discussion of how shifts in \( d \) may affect the dynamics in the real sector and possibly, though more indirectly, the intensity of herding.

8. The impact of changes in financial distress

The model set out above provides a framework that, without going into the specification details of, for example, the banking system, allows us to study how a financial crisis may spill over to the dynamics of the real sector. Furthermore, these real effects may in turn change some of the parameters in the present model, especially those governing the adjustments of the business sentiment. It is nevertheless useful to remain in our two-dimensional framework and begin this discussion with \textit{ceteris paribus} variations of the financial distress parameter \( d \) and subsequently two other parameters in the switching index, whose impact on the trajectories in this economy has then to be examined. A full-fledged investigation that endogenizes the parameter changes and possibly also some

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26 Incidentally, this diagram is similar to Figure 2a in the important paper by Lux (1995, p. 888), where he integrated the transition probability approach and its herding component into an asset pricing model. He also briefly mentioned (on p. 889) the possible occurrence of the other stability or instability phenomena dealt with in Figure 3.
feedbacks from the real to the financial sector may be undertaken in a later stage of research. We expect that their results would be better understood on the basis of what we observe in the following phase diagram analysis.

<table>
<thead>
<tr>
<th></th>
<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
<th>S7</th>
</tr>
</thead>
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<td>$d$</td>
<td>0.00</td>
<td>0.15</td>
<td>0.20</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.10</td>
</tr>
<tr>
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<td>1.20</td>
<td>1.20</td>
<td>1.20</td>
<td>0.50</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>$\phi_y$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.13</td>
<td>1.00</td>
</tr>
<tr>
<td>$\phi_x + \eta \phi_y$</td>
<td>6.70</td>
<td>6.70</td>
<td>6.70</td>
<td>6.70</td>
<td>6.00</td>
<td>6.72</td>
<td>6.00</td>
</tr>
<tr>
<td>$\gamma^{e=\text{equil}}$</td>
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<td>-2.01</td>
<td>-2.76</td>
<td>-1.35</td>
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<td>$\pi^{e=\text{equil}}$</td>
<td>2.50</td>
<td>1.87</td>
<td>1.70</td>
<td>1.40</td>
<td>1.96</td>
<td>1.38</td>
<td>2.32</td>
</tr>
</tbody>
</table>

Table 2: Parameter scenarios S1–S7.

Let us start from the benchmark scenario of Section 6 with its repelling steady state position and the unique and globally attractive limit cycle, which characterizes the economy under normal conditions $d=0$ on the financial markets. However, in the sentiment switching index $s$ we now want to discuss a possible role for the accelerator parameter $\phi_y$, which was previously neglected and set equal to zero. We thus assign a positive value to it and simultaneously decrease the herding parameter $\phi_x$ such that the original level $\phi_x + \eta \phi_y = 6.70$ is maintained. Calling this parameter scenario S1, the precise values are shown in the second column of Table 2. For convenience, the isoclines of S1 in the phase plane and the limit cycle are reproduced in the top-left panel of Figure 4.

Being on or close to the closed orbit of S1, suppose a crisis starts within the financial sector and then in some way affects the general state of confidence in the real sector. Accordingly, the distress parameter $d$ will be gradually increased. As is clear from eqs. (6) and (8), this leaves the IC($\pi^c$) isocline intact, while it shifts IC($\pi^c$) downward. As a consequence, the point of intersection of the two curves slides downward on the IC($\pi^c$) isocline, which will also have a bearing on the stability properties of this equilibrium point.

Before going on, it should be pointed out that similar situations to the ones we will consider would also arise if we started out from a scenario with multiple equilibria, as in the last four panels in Figure 3. The downward shift of the $\dot{x} = 0$ isocline will soon cause the inner and high-activity equilibria to disappear, so that there would be only one equilibrium left with a more or less distinct dominance of pessimistic firms.

Since in scenario S1 the slope of the IC($\pi^c$) isocline is steepest at $x=0$, the downward shift of the equilibrium that is brought about by the increasing $d$ is associated with a decline of the slope of IC($\pi^c$) in this point. If the increase is limited, the slope will not only remain positive but this state may also continue to be repelling, so that the economy is
Figure 4: The dynamics of scenario S1–S7 from Table 2.

again characterized by a globally attractive limit cycle. These features prevail in the case \( d = 0.15 \) in scenario S2. The new limit cycle is shown as the bold (red) line in the upper-right panel in Figure 4. The solid (green) line repeats the limit cycle from the previous panel, from which it is seen that the cyclical motion has shifted to the south-west and has also become narrower. The third column in Table 2 reports that the output gap oscillates around an equilibrium value of \(-1.57\%\), and the inflation rate around 1.87\% (instead of the target value of 2.50\%).

Furthermore, the shift in \( d \) brings the nonlinear nature of the dynamic system more clearly to the fore. It may in fact be observed that these oscillations are no longer symmetrical as in S1; rather, the upper turning points of \( x, \pi^c \), and thus also of \( y, \pi \), deviate more from the equilibrium values than the lower turning points. In particular, the troughs and peaks of the output gap are (in per cent) \(-3.50 = -1.57 - 1.93 \) and \( 0.90 = -1.57 + 2.47 \), respectively.

We thus see that if hypothetically, after the onset of the financial crisis, \( d \) did not change for a longer while, the economy would be in a cyclical regime of, on average,
moderately reduced economy activity.

By contrast, scenario S3 supposes a subsequent increase of financial distress from \( d = 0.15 \) to \( d = 0.20 \). The main effect, illustrated in the middle-left panel in Figure 4, is that the instability of the equilibrium turns into stability, where for all practical purposes the attractiveness is global. Convergence is of a cyclical nature and relatively slow. It takes several cycles, each of which with a period of roughly 7.70 years, until the oscillations die out. The scenario is therefore qualitatively not so much different from the cyclical behaviour in S2, in particular, if it is taken into account that realistically these motions were disturbed by stochastic noise. Quantitatively, of course, inflation and economic activity have declined further.

A stronger aggravation of the financial crisis and its consciousness in the real sector, which may be represented by \( d = 0.30 \) in scenario S4, invigorates the gravitation forces of the equilibrium point. In this way, the cyclical tendencies almost disappear, which makes convergence considerably faster. On the sample trajectory shown in the middle-right panel in Figure 4, it takes about 6.40 years from its starting point until convergence is practically completed when the business sentiment hits the \( IC(x) \) isocline for the second time. From then on, the economy would be trapped in a recessionary state of equilibrium with an output gap around \( y = -2.76\% \) and a rate of inflation that is almost one percentage point below the central bank’s target.

Certainly, financial distress will not stay constant all the time until the equilibrium state will be reached. For the sake of the argument, let us nevertheless ask if in such a hypothetical situation there are prospects, besides massive government intervention, to alleviate it. With respect to the equilibrium point shown in the middle-right panel, it is our idea that after the economy has essentially come to rest the agents will to some extent change their behavioural patterns, where in the first instance we think of the fixed investment of firms.

Consider that in the assumed recession there is a clear majority of pessimistic firms and everyone has learned about its obstinate persistence. On the other hand, while each firm is hoping for the light at the end of the tunnel, it also wants to be sure that this is not an optical illusion. Knowing the high risk of moving too early, i.e. of switching too early from pessimism to optimism, firms will now require the incoming good news to be more substantial if it is to increase their capital growth rate. Correspondingly, the fear of falling behind plays a smaller role, so that they will less easily suppress their own beliefs and will less easily base their long-term investment on recent changes in the attitudes of other firms. Therefore, the weights between self-reference and hetero-reference in the switching mechanism in our theoretical framework may shift in favour of hetero-reference (see the discussion in the second half of Section 2 for these concepts). Specifically, the firms may be more attentive to changes in the real interest rate as a positive signal than to possible movements in the average sentiment. This idea is conveniently represented by a reduction of the herding parameter \( \phi_x \) (and no change in the other coefficients for
greater clarity).

It is evident that this is a psychological reasoning. Although the role of psychology in economic behaviour is now widely acknowledged, it could be generally objected that, *a priori*, statements to the contrary of a plausible psychological hypothesis might appear similarly plausible. A certain support of our hypothesis of reduced herding in the recession may—by way of analogy—be seen in the empirical studies by Christie and Huang (1995) and Gleason et al. (2004) for American stock markets. Covering price data at different frequencies (intraday, daily and monthly), their main conclusion is that herding of investors is not an important factor during periods of market stress. Moreover, these episodes would rather tend to corroborate the predictions of rational asset pricing models.\(^{27}\)

Scenario S5 maintains the level of financial distress and considers a *ceteris paribus* decrease of \(\phi_x\) from 1.20 to 0.50.\(^{28}\) The point \((x, h_x(x, d)) = (0, h_x(0, 0.30))\) on the IC\((x)\) isocline in the phase plane remains a fixed anchor, but the resulting decrease of the composite coefficient \(A\) from 1.387 to 0.687 causes the positive slope of IC\((x)\) at this point to become negative. Since the slope even decreases globally, at each point \(x\) as it is immediately seen from (9), the point of intersection of IC\((x)\) and IC\((\pi^c)\) slides upwards on the latter isocline. From Proposition 3.2 we know that this equilibrium, too, is globally attractive.

The bottom-left panel in Figure 4 illustrates a transition from a position near the old towards the new equilibrium. The trajectory starts at \(y = -3.43\%\) and needs 1.12 years to find back to normal utilization, when \(y = 0\%\). The upswing then eases off, such that in the next (almost) six months the output gap continues its rise to a peak of 0.25\% only. The ensuing decline towards the equilibrium output gap of \(-1.35\%\) (see Table 2) takes somewhat more than two years. Thus, on the whole, less intense herding is able to weaken the consequences of the financial crisis and bring about a mild, though not complete recovery of the economy.

Even if the fall of \(\phi_x\) is accepted as a plausible change in the investment behaviour of firms in a recession, scenario S5 is not the entire story that could be told. Regarding

\(^{27}\) Market stress in these studies is characterized by strong price changes. However, a recession is not a tranquil state, either, where firms would reduce activity and just hibernate. At the micro level there will rather be a large number of dramatic and often contradictory events, which may be as confusing as the volatile price movements on the individual stock markets.

\(^{28}\) Behavioural parameters in a model are usually treated as fixed magnitudes. An exception is Lux (1995, Section III). Starting from the same question in his asset pricing model as we, how can the agents escape an extreme (bull or bear market) equilibrium, he discusses changes of a predisposition coefficient which is supposed to represent a general confidence in the market and would here be another constant in the specification (SI) of the switching index. Lux, then, endogenizes this coefficient and lets the current asset returns feed back on it such that the bubble equilibria disappear. Similar or analogous ideas might also be considered for our herding parameter \(\phi_x\), though an endogeneization of the financial distress variable \(d\) should have a higher priority. Both kinds of generalization are, however, beyond the scope of the present paper.
the relatively greater weight of hetero-reference in the switching mechanism, it is also conceivable that the firms become more responsive to the observed changes in overall economic activity. Accordingly, the reduction of $\phi_x$ may be accompanied by an increase of the accelerator coefficient $\phi_y$, which has the opposite effect on the central parameter $A$. With S6, Table 2 presents a scenario where such an increase of $\phi_y$ just undoes the previous decrease of $A$. The dynamics is thus (approximately) equal to scenario S4.

If, for example, firms attached a higher weight to the accelerator argument a few months after they have reduced herding, the beginning upturn of scenario S5 would be scotched straightaway and after a short intermezzo the economy would return to a low-activity equilibrium with $y \approx -2.8\%$. Remarkably, the reason for this relapse would be hard to identify by the economic experts observing the economy; it would possibly be sought elsewhere.

To complete the discussion of the spillover from the financial sector, scenario S7 in Table 2 is put forward as the most optimistic story line (or wishful thinking, it might be said). It neglects the possible effects from a rise of $\phi_y$, resetting this coefficient to its previous value $\phi_y = 1.00$, maintains the reduction of herding ($\phi_x = 0.50$), and assumes that the worst is considered to be over on the financial markets, which is captured by a decline of $d$ back to almost normal, $d = 0.10$ say.

The properties of this dynamic system are by now clear enough. The equilibrium point moves up the IC($\pi^c$) isocline, and it is globally stable. As the bottom-right panel in Figure 4 shows, its cyclical tendencies are fairly weak, that is, in the course of the recovery there is one positive overshooting in the output gap, after which convergence practically occurs in a monotonic way. This is a time where herding might regain its importance and $\phi_x$ might rise again to shift the economy into a cyclical regime similar to the ones we have begun with.

9. Conclusion

This paper has advanced a modelling strategy to devise the dynamic adjustments of a general sentiment index and establish a sound microfoundation for it, which is fundamentally different from the ruling paradigm of the representative agent where micro and macro tend to collapse into the same thing. The present approach is rather based on individual agents that randomly switch between, in its simplest form, two modes of behaviour, where the index is an aggregate measure of these modes in the total population. This constitutes a truly dynamic framework since the transition probabilities, and thus the change in the index, are dependent on the current state of the economy as well as on the current composition of the population itself; and certainly the variations in the index change the aggregate actions of the agents and thus again the state of the economy.

For a general characterization of the resulting dynamic feedbacks, the following three features are worth pointing out. (1) Even in a macroeconomic state of rest, there are still
permanent switches going on at the micro level. (2) The adjustments of the index contain a mechanism of global self-stabilization with reflecting boundaries. (3) The specification of the macro equation governing the variations of the sentiment index is particularly well suited to incorporate a herding component, such that varying strengths of it can give rise to quite different dynamic scenarios.

The approach can be conveniently applied to a great variety of decision problems in economics. Here we focussed on an interpretation of the index variable as a general business sentiment determining fixed investment, which can give the Keynesian notion of “animal spirits” a more precise and tractable meaning, and which we considered to be more meaningful than the omnipresent expectations of a few macroeconomic variables in the next period. It was furthermore obvious that our approach is highly flexible in that the transition probabilities can be made dependent on all sorts of feedbacks. The concept of the associated sentiment adjustment equation could therefore be easily integrated into many models from the (heterodox) macroeconomic literature that might wish to discuss something like the animal spirits.

The notion of a dynamic business sentiment variable was then used to set up an alternative, so to speak old-Keynesian version of the new macroeconomic consensus with its three equations for output, inflation, and the interest rate. In more detail, we formulated a few simplifying assumptions to the effect that the motions of the output gap are directly linked to the business sentiment. The concept of an adaptive inflation climate that gradually adjusts towards a weighted average of current and target inflation was introduced into an otherwise ordinary Phillips curve. And third, the rate of interest was determined by a Taylor rule. Since the rate feeds back on the business sentiment, it may thus counteract the herding tendencies mentioned above. In fact, these mechanisms are at the heart of the endogenous persistent cycles that we found, or the centrifugal and centripetal forces neutralize each other in a way that lads to two additional equilibria, besides the steady state, with high or low economic activity, respectively. We may claim that the present modelling framework provides a parsimonious and elegant specification to generate, and to understand, these dynamic phenomena in a nonlinear two-dimensional differential equations system.

Future research may go into three directions. First, it may be devoted to combining the business sentiment adjustments with elements from other and more encompassing models in the (heterodox) macroeconomic literature. Thus, there may be a role for a time-varying income distribution to add a Goodwinian flavour to the present model. Or one borrows a financial sector from the literature with the aim to turn our shift parameter $d$, which we called ‘financial distress’, into an endogenous variable.

29 For example, one might take the Keynes-Goodwin model of Franke and Asada (1994), replace the ad-hoc adjustment equation for its ‘state of confidence’ with the specification of the present sentiment variable (in essence, the two are actually not so much different), and replace the LM part with the present Taylor rule.
Second, more than two attitudes may be admitted between which the agents can choose. Explorations with a three-state version of the Goodwinian, though in its mathematical structure very similar, model by Franke (2008a) indicate that even with strong herding, two different regimes might then coexist: one exhibiting persistent and bounded fluctuations similar to those in this paper, and one where a second long-run equilibrium position (with a higher share of neutral agents) proves to be locally attractive.

A third task is an econometric estimation of either the full model put forward here or, perhaps to begin with, of its two main building blocks: the business sentiment dynamics and the adaptive inflation climate in the Phillips curve. In the light of our preliminary numerical calibration of the cyclical scenario, we believe that the method of simulated moments should be a fruitful approach to this endeavour.

Mathematical Appendix

Proof of Proposition 1.
The first part of the proposition follows from an application of the Implicit Function Theorem to the identity \( F_x[x, h_x(x, d)] = 0 \) for the \( \dot{x} = 0 \) isocline in (10), from which, with the notation of eq. (11), we get \( \partial h_x/\partial x = -F_{xx}/F_{x\pi} = F_{xx}/F_{x\pi} \). Part 2 is then a direct consequence of the remark on (11). Local stability in the third part follows from the fact that \( A < 1 \) entails a negative square bracket in (9). The latter even holds true globally, so that, as already noted, the \( \dot{x} = 0 \) isocline is everywhere downward-sloping. Hence it can have only one point of intersection with the everywhere rising \( \dot{\pi}^c = 0 \) isocline, which proves the uniqueness of the stationary point. Its global stability can be concluded from Olech’s Theorem (see, e.g., Gandolfo, 1997, pp. 354f): as it is immediately checked, on the entire domain of the system the trace of the Jacobian is negative, the determinant is positive, and the two off-diagonal entries remain both nonzero. Finally, eq. (8) shows that the \( \dot{x} = 0 \) isocline intersects the (non-horizontal) \( \dot{\pi}^c = 0 \) isocline at \( x = 0, \pi^c = 0 \) if and only if \( d = 0 \).

q.e.d.

Proof of Proposition 2.
The statement about the relative slopes of the isoclines says that \( \partial h_x/\partial x > d\pi/\partial x \). In the proof of the previous proposition, the Implicit Function Theorem was already used to establish that \( \partial h_x/\partial x = -F_{xx}/F_{x\pi} \). By the same token, \( d\pi/\partial x = -F_{xx}/F_{x\pi} \). Taking the signs of the partial derivatives into account, \( \partial h_x/\partial x > d\pi/\partial x \) is easily seen to be equivalent to \( \det J < 0 \), which characterizes a saddle point.

q.e.d.

Proof of Proposition 3.
\((x, \pi^c) = (0, \pi^*)\) is the only point of rest if \( A < 1 \), which is already known from Proposi-
tion 1.3, or if, with $A \geq 1$, the slope of the $\dot{x} = 0$ isocline is nonnegative at that point but less than that of the $\dot{\pi}^c = 0$ isocline, i.e., if $\partial h_x(0)/\partial x \leq dh_\pi(0)/dx$. The latter follows from eq. (9), according to which $\partial h_x(x)/\partial x$ decreases as $|x|$ increases, whereas additional points of intersection of the two isoclines would require it to increase eventually. By the same argument as in the proof of Proposition 2, the inequality relationship of the two slopes is equivalent to $\det J \geq 0$, and this is equivalent to the inequality stated in the first part of the proposition (recalling that $\cosh(s) = 1$ in the steady state).

If, on the other hand, this inequality is violated so that $\det J < 0$, the $\dot{x} = 0$ isocline cuts the $\dot{\pi}^c = 0$ isocline from below. Since the second isocline is an increasing straight line and the first one tends to $\infty$ as $x \to -1$ from the right, and to $-\infty$ as $x \to 1$ from the left, there will be two additional points of intersection.

Regarding the second part, the inequality $A < 1 + \alpha \gamma / 2 \nu$ is equivalent to $\text{trace} J < 0$ when the Jacobian is evaluated at the steady state, while it has just been established that $\det J > 0$ if and only if $A < 1 + (1-\gamma) B / \gamma$. Outside the steady state, the upper-diagonal entry of $J$ can only be smaller, so that $\text{trace} J < 0$ as well as $\det J > 0$ are maintained. Since the off-diagonal elements are nonzero in any case, Olech’s Theorem again ensures global asymptotic stability of the steady state.

According to the reasoning above, the two inequalities in Proposition 3.3 are equivalent to $\det J > 0$ and $\text{trace} J > 0$ in the equilibrium point, which is therefore repelling. The condition on $B$ is a mere restatement of the condition that the right-hand side of the two inequalities exceeds the left-hand side. Finally, a Hopf bifurcation occurs at $A = 1 + \alpha \gamma / 2 \nu$ since here the determinant of $J$ is positive.

$q.e.d.$

**Proof of Proposition 5.**

Since $\text{IC}(\pi^c)$ is a straight line and by virtue of (9) $\partial h_x(x)/\partial x$, i.e. the slope of $\text{IC}(x)$, decreases as $|x|$ increases, there can be only one point to the right of $x = 0$ where the two isoclines intersect, and one to the left. The symmetry of $x^L$ and $x^H$ is obvious from the symmetry of the two isoclines with respect to the steady state.

$q.e.d.$

**References**


