Multifractality and Long-Range Dependence of Asset Returns: The Scaling Behaviour of the Markov-Switching Multifractal Model with Lognormal Volatility Components

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Abstract

In this paper we consider daily financial data from various sources (stock market indices, foreign exchange rates and bonds) and analyze their multi-scaling properties by estimating the parameters of a Markov-switching multifractal model (MSM) with Lognormal volatility components. In order to see how well estimated models capture the temporal dependency of the empirical data, we estimate and compare (generalized) Hurst exponents for both empirical data and simulated MSM models. In general, the Lognormal MSM models generate 'apparent' long memory in good agreement with empirical scaling provided one uses sufficiently many volatility components. In comparison with a Binomial MSM specification [7], results are almost identical. This suggests that a parsimonious discrete specification is flexible enough and the gain from adopting the continuous Lognormal distribution is very limited.

Keywords: Markov-switching multifractal; scaling; return volatility.

1 Introduction

The development of the multifractal approach goes back to Benoit Mandelbrot's work on turbulent processes in statistical physics [12]. Its adaptation for financial data resulted in the multi-fractal model of asset returns (MMAR) [13] which provides a new time series model with attractive stochastic properties accounting for the stylized facts of financial markets. However, the practical applicability of MMAR suffers from its combinatorial nature and from its non-stationarity due to the restriction to a bounded interval; in addition, it suffers from a lack of applicable statistical methods, see [13, 9]. These limitations have been overcome by the introduction of iterative versions of multifractal processes [3, 10] which preserve the multifractal and stochastic properties but have more convenient asymptotical properties.

In this paper, we expand on our previous paper [7] and compare the scaling properties of the empirical data to those of estimated Markov-switching multifractal models. Based on the empirical estimates via Generalized Method of Moments (GMM), simulations are conducted, and we compare the empirical data and simulated ones in terms of their autocorrelation functions (ACF). In addition, we compute the generalized Hurst exponent $H(q)$ by means of the modified R/S method [8] and the approach developed in [4, 5, 6]. We proceed by comparing the scaling exponents for empirical data and simulated time series based on our estimated MSM models. The structure of the paper is as follows: In Section 2 we introduce the multifractal models. Section 3 reports the empirical and simulation-based results. A summary and concluding remarks are given in Section 4.

2 Markov-switching multifractal models

In the Markov-switching multifractal model, financial asset returns are modelled as:

$$r_t = \sigma_t \cdot u_t \tag{1}$$

with innovations u_t drawn from the standard Normal distribution $N(0,1)$ and instantaneous volatility being determined by the product of k volatility components or multipliers $M_t^{(1)}$, $M_t^{(2)}$ ..., $M_t^{(k)}$ and a constant scale factor σ:

$$\sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^{(i)}. \tag{2}$$

Each volatility component is renewed at time t with probability γ_i depending on its rank within the hierarchy of multipliers and it remains unchanged with probability $1-\gamma_i$. The transition probabilities are specified by Calvet and Fisher [3] as:

$$\gamma_i = 1 - (1-\gamma_k)^{(b^{i-k})} \qquad i = 1, \ldots k, \tag{3}$$

with parameters $\gamma_k \in [0,1]$ and $b \in (1,\infty)$. Different specifications of Eq. (3) have been imposed (cf. [10] and its earlier versions). By fixing $b = 2$ and $\gamma_k = 0.5$, we arrive at a relatively parsimonious specification:

$$\gamma_i = 1 - (\frac{1}{2})^{(2^{i-k})} \qquad i = 1, \ldots k. \tag{4}$$

For the choice of volatility components, a popular version of the MF process adopts the Binomial distribution: $M_t^{(i)} \sim \{m_0,\ 2 - m_0\}$ with $1 \leq m_0 < 2$. Another prominent variant [13, 10] is the continuous version of the multi-fractal process that assumes the volatility components to be random draws from a Lognormal distribution[1] (LN) with parameters λ and σ_m, i.e.

$$M_t^{(i)} \sim LN(-\lambda, \sigma_m^2). \tag{5}$$

In line with the combinatorial settings (cf. [13, 2]) a normalisation of the expectation value of $M_t^{(i)}$, that is, $E[M_t^{(i)}] = 1$ is imposed which leads to a restriction on the paramenters of the Lognormal distribution:

$$exp(-\lambda + 0.5\sigma_m^2) = 1 \qquad \Rightarrow \qquad \sigma_m = \sqrt{2\lambda}. \tag{6}$$

Note that the admissible parameter space for the location parameter λ is $[0, \infty)$ where in the borderline case $\lambda = 0$ the volatility process collapses to a constant (the same when $m_0 = 1$ in the Binomial case).

The above multi-fractal processes can be viewed as a special case of a Markov-switching process which makes maximum likelihood (ML) estimation feasible if the distribution of volatility components is discrete. In the Binomial case, state spaces are finite, so that maximum likelihood estimation is possible, cf. Calvet and Fisher (2004) [3]. However, the applicability of ML encounters an upper bound for the number of cascade levels (about $k \leq 10$) because of the necessity to evaluate the $2^k \times 2^k$ transition matrix for every realization. The limits of current computational capability are reached with about 10 cascade levels. A more fundamental limitation is the restriction to cases that have discrete distributions of volatility components. Since multifractal processes with continuous distributions (such as the Lognormal distribution) of the volatility components imply an infinite number of states, maximum likelihood is not applicable to them. Lux [10] proposed the Generalized Method of Moments (GMM) approach as an alternative, which relaxes these computational restrictions. GMM is typically applicable without computational restrictions and can be used in the case of the Binomial MF model for larger numbers of cascade levels ($k > 10$), and the Lognormal MF process. The analytical moment conditions for implementing GMM (both the Binomial and the Lognormal model) can be found in [10].

Using the iterative version of the multifractal model instead of its combinatorial predecessor and confining attention to unit time intervals, the resulting dynamics of eq. (1) can also be viewed as a particular version of a stochastic volatility model. With this rather parsimonious approach, this pertinent MF process nevertheless preserves the hierarchical structure of MMAR while dispensing with its restriction to a bounded interval. The model also captures some properties of financial time series, namely, outliers (extreme realizations),

[1] The lognormal distribution has the probability density function $f(x, \lambda, \sigma_m) = \frac{e^{-(lnx-\lambda)/(2\sigma_m^2)}}{x \cdot \sigma_m \sqrt{2\pi}}$.

volatility clustering ([11]) and the power-law behaviour of the autocovariance function:

$$\langle (|r_t|^q - \langle |r_t|^q \rangle) \cdot (|r_{t+\tau}|^q - \langle |r_{t+\tau}|^q \rangle) \rangle \propto \tau^{2d(q)-1}, \quad (7)$$

where for each qth moment and time lag τ, $d(q)$ is the pertinent scaling function depending on q (for the detailed proof, cf. [3]). Although models of this class are partially motivated by empirical findings of long-term dependence of volatility, they do, however, not obey the traditional definition of long memory, i.e. asymptotic power-law behavior of autocovariance functions in the limit $t \to \infty$ or divergence of the spectral density at zero, see [1]. The iterative MF model is rather characterized by only 'apparent' long memory with an asymptotic hyperbolic decline of the autocorrelation of absolute powers over a finite horizon and exponential decline thereafter. In the case of Markov-switching multifractal process, the approximately hyperbolic decline, therefore, holds only over an interval $1 \ll \tau \ll 2^k$.

3 Comparison of empirical and simulated series

In this paper, we consider daily data for a collection of stock exchange indices: the Dow Jones Composite 65 Average Index (Dow) and $NIKKEI$ 225 Average Index (Nik) over the time period from January 1969 to October 2004, foreign exchange rates: British Pound to U.S. Dollar (UK), and Australian Dollar to U.S. Dollar (AU) over the period from March 1973 to February 2004, and U.S. 1 year and 2 years treasury constant maturity bond rates ($TB1$ and $TB2$, respectively) over the period from June 1976 to October 2004. The daily prices are denoted as p_t, and returns are calculated as $r_t = \ln(p_t) - \ln(p_{t-1})$ for stock indices and foreign exchange rates and as $r_t = p_t - p_{t-1}$ for $TB1$ and $TB2$.

We estimate the Lognormal model parameters via Generalized Method of Moments (GMM). Table 1 presents the empirical estimates of the Lognormal model for various hypothetical numbers of cascade levels ($k = 5, 10, 15, 20$) using the same analytical moments as in [10] (numbers within the parentheses are the standard errors). The pertinent estimates for the Binomial case have been reported in [7]. For each time series, we find that the estimates for $k \geq 10$ are almost identical. In fact, analytical moment conditions in [10] show that higher cascade levels make a smaller and smaller contribution to the moments so that their numerical values would stay almost constant. If one monitors the development of estimated parameters with increasing k, one finds strong variations initially with a pronounced decrease of the estimates which become slower and slower until, eventually a constant value is reached somewhere around $k = 10$ for each time series.

As a prelude to our Monte Carlo study comparing the empirical and simulated scaling exponents, we plot the autocorrelation functions (ACF) for empirical and simulated time series with different k values (Figure 1). We find that the simulated time series with $k = 5$ exhibits much faster decay than the empirical data. In contrast, the ones with larger values of k show the ability of the MSM

Table 1: GMM estimates of MSM model for different values of k.

	k = 5		k = 10		k = 15		k = 20	
	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\sigma}$
Dow	0.148	0.983	0.139	0983	0.139	0.983	0.139	0.983
	(0.018)	(0.052)	(0.018)	(0.052)	(0.018)	(0.053)	(0.018)	(0.052)
Nik	0.289	0.991	0.279	0.990	0.280	0.991	0.280	0.990
	(0.022)	(0.036)	(0.021)	(0.036)	(0.022)	(0.036)	(0.022)	(0.036)
UK	0.096	1.053	0.079	1.058	0.078	1.058	0.078	1.058
	(0.018)	(0.027)	(0.017)	(0.026)	(0.017)	(0.027)	(0.017)	(0.027)
AU	0.140	1.012	0.121	1.014	0.120	1.015	0.120	1.014
	(0.024)	(0.065)	(0.023)	(0.065)	(0.023)	(0.066)	(0.023)	(0.065)
*TB*1	0.315	1.049	0.276	1.075	0.275	1.077	0.275	1.077
	(0.021)	(0.061)	(0.019)	(0.061)	(0.019)	(0.061)	(0.019)	(0.061)
*TB*2	0.469	1.013	0.403	1.047	0.401	1.048	0.401	1.049
	(0.022)	(0.056)	(0.019)	(0.054)	(0.019)	(0.054)	(0.019)	(0.054)

Note: Estimation is based on the Lognormal model. All data have been standardized before estimation.

model to replicate the empirical autocorrelation function, namely, the hyperbolic decay of ACF. By recalling eq. (7), we recognize that the approximately hyperbolic decline holds over an interval $1 \ll \tau \ll 2^k$, therefore, a multifractal process with a higher number of cascade levels implies a longer power-law range of the autocorrelations, which means a larger region of apparent long-term dependence. We have also studied the ACFs based on the Binomial model, and they show pretty similar patterns. Studies on other time series have also been pursued; we omit them here as they are qualitatively similar to the one of the Dow Jones index.

We have computed $H(q)$ by means of the generalized Hurst exponent (GHE) approach [4, 5, 6] and H by means of the modified R/S method [8] for the same data sets, and we proceed by comparing the scaling exponents obtained for empirical data and simulated time series based on the estimated Lognormal MSM models. For the results reported in Table 2, we focus on $H(q)$ for $q = 1$ and $q = 2$ for the empirical time series as well as for 1000 simulated time series of each set of estimated parameters. The values for $H(1)$ and $H(2)$ are averages computed from a set of scaling exponents corresponding to different $\tau_{\max}$ (between 5 and 19 days) for the stochastic variable $X(t)$ in ([4, 5, 6]) defined as the sum of absolute value of returns $X(t) = \sum_{t'=1}^{t} |r_{t'}|$. The second and seventh columns in Table 2 report the empirical $H(1)$ and $H(2)$, and values in the other columns are the mean values over the corresponding 1000 simulations for different k values: $5, 10, 15, 20$, with errors given by their standard deviations. Boldface numbers are those cases which fail to reject the null hypothesis that the mean of the simulation-based Generalized Hurst exponent values equals the

Table 2: H(1) and H(2) for the empirical and simulated data.

	$H(1)$					$H(2)$				
	Emp	$k=5$	$k=10$	$k=15$	$k=20$	*Emp*	$k=5$	$k=10$	$k=15$	$k=20$
DOW	0.684	0.748	0.847	0.867	0.866	0.709	**0.710**	0.797	0.812	0.810
	(0.034)	(0.010)	(0.017)	(0.023)	(0.025)	(0.027)	(0.011)	(0.017)	(0.020)	(0.021)
AU	0.827	0.742	**0.836**	**0.857**	**0.857**	0.722	**0.706**	0.790	0.806	0.807
	(0.017)	(0.010)	(0.019)	(0.024)	(0.025)	(0.024)	(0.012)	(0.019)	(0.021)	(0.022)
TB1	0.853	**0.857**	0.908	0.912	0.910	0.814	**0.782**	**0.823**	**0.824**	**0.823**
	(0.022)	(0.035)	(0.026)	(0.027)	(0.028)	(0.027)	(0.027)	(0.023)	(0.024)	(0.023)

Note: *Emp* refers to the empirical estimates of $H(1)$ and $H(2)$. $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the mean and standard deviation of the exponent values based on 1000 simulated time series with pertinent k (Lognormal model). Bold numbers show those cases for which we cannot reject identity of the Hurst coefficients obtained for empirical and simulated data, i.e. the empirical exponents fall into the range between the 2.5 to 97.5 percent quantile of the simulated data.

empirical Generalized Hurst exponent at the 5% level based on the distribution of our 1000 Monte Carlo samples. We find that the exponents from the simulated time series vary across different cascade levels k. In particular, for the stock market indices, we find coincidence between the empirical series and simulation results for the scaling exponents $H(2)$ for the Dow Jones index. For the exchange rate data, we observe that the simulations successfully replicate the empirical measurements of AU for $H(1)$ when $k = 10, 15, 20$. In the case of U.S. Bond rates, we find a good agreement for $H(1)$ when $k = 5$ and for all k for $H(2)$. While the empirical numbers are in nice agreement with previous results in [4, 5, 6], it is interesting to note that simulated data with $k \geq 10$ have a tendency towards even higher estimated Hurst coefficients than found in the pertinent empirical records. Similar studies for the Binomial model as well as GHE for other specifications of the stochastic variable $X(t)$ can be found in [7].

To assess the ability to replicate empirical scaling behaviour, we also performed calculations using the modified Rescaled Range (R/S) analysis introduced by Lo (1991) [8], whose results are reported in Tables 3 to 5. Table 3 presents Lo's test statistics for both empirical and 1000 simulated time series (absolute returns) based on the Lognormal model with different values of k and for different truncation lags $\tau = 0, 5, 10, 25, 50, 100$. We find that the values are varying with different truncation lags, and more specifically, that they are monotonically decreasing for both the empirical and simulation based statistics. Table 4 reports the number of rejections of the null hypothesis of short-range dependence based on 95% and 99% confidence levels. The rejection numbers for each single k are decreasing as the truncation lag τ increases, but the proportion of rejections remains relatively high for higher cascade levels, $k = 10, 15, 20$. The modified R/S approach would quite reliably reject the null of long memory for $k = 5$, but in most cases it would be unable to do so for higher numbers of volatility components, even if we allow for large truncation lags up to $\tau = 100$.

This appears to be harmony with the impression conveyed by Figure 1. The corresponding Hurst exponents are given in Table 5. The empirical values of H are decreasing when τ increases, and a similar behaviour is observed for the simulation-based H for given values of k. We also find that the Hurst exponent values are increasing with increasing cascade level k for given τ. Boldface numbers are those cases which fail to reject the null hypothesis that the mean of the simulation-based Hurst exponent equals the empirical Hurst exponent at the 5% level, and we observe similar scenarios for the pertinent results based on the Binomial model reported in [7]. There are significant jumps between the values for $k = 5$ and $k = 10$ as in previous tables, and we observe a good overall agreement between the empirical and simulated data for practically all series for $k \geq 10$, but not so for the MSM models with smaller number of volatility components, e.g. $k = 5$.

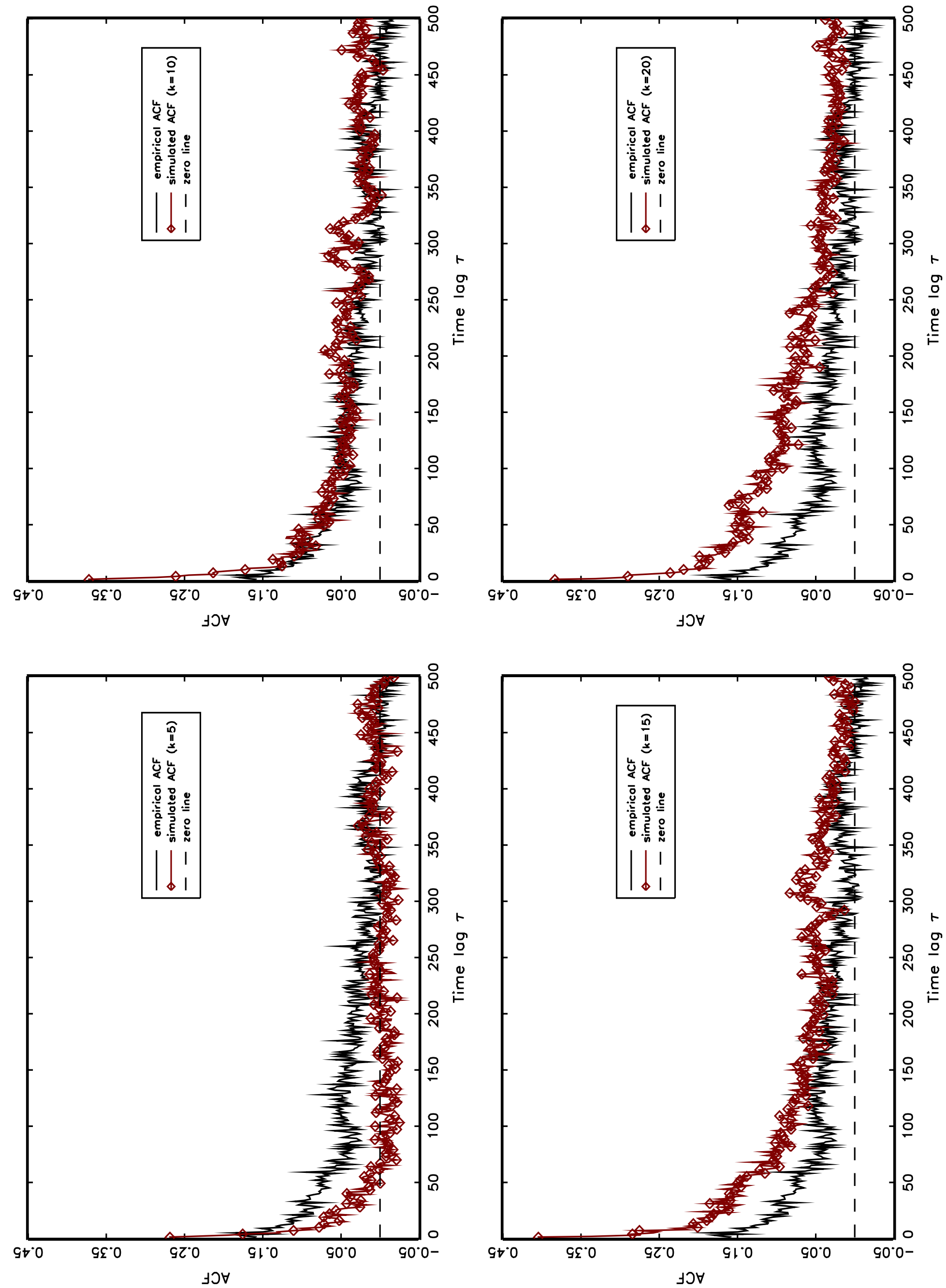

Figure 1: Autocorrelation function (ACF) for the Dow Jones index and simulated time series (absolute returns). All simulations are based on the Lognormal model with different k values.

Table 3: Lo's R/S statistic for the empirical and simulated data.

	$\tau = 0$					$\tau = 5$					$\tau = 10$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	3.005	1.710	4.959	6.511	6.515	2.661	1.475	3.956	5.025	5.116	2.427	1.369	3.484	4.380	4.459
		(0.380)	(1.316)	(1.773)	(1.830)		(0.327)	(1.021)	(1.338)	(1.375)		(0.302)	(0.884)	(1.140)	(1.167)
Nik	7.698	1.837	4.633	5.809	5.802	6.509	1.530	3.616	4.495	4.487	5.836	1.405	3.175	3.925	3.917
		(0.428)	(1.207)	(1.543)	(1.528)		(0.352)	(0.926)	(1.168)	(1.157)		(0.321)	(0.802)	(1.004)	(0.996)
UK	6.821	1.570	4.480	5.995	6.035	5.912	1.390	3.712	4.874	4.900	5.333	1.304	3.313	4.298	4.319
		(0.377)	(1.237)	(1.837)	(1.842)		(0.332)	(0.988)	(1.410)	(1.417)		(0.309)	(0.863)	(1.202)	(1.211)
AU	7.698	1.670	4.832	6.247	6.294	6.731	1.443	3.891	4.941	4.975	6.103	1.341	3.436	4.316	4.344
		(0.380)	(1.300)	(1.852)	(1.818)		(0.325)	(1.017)	(1.401)	(1.366)		(0.300)	(0.882)	(1.192)	(1.159)
*TB*1	8.845	1.812	4.485	5.467	5.431	7.109	1.512	3.509	4.245	4.212	6.110	1.390	3.082	3.708	3.676
		(0.428)	(1.141)	(1.432)	(1.405)		(0.353)	(0.873)	(1.080)	(1.064)		(0.321)	(0.755)	(0.925)	(0.913)
*TB*2	7.295	1.811	4.202	4.984	4.892	6.083	1.496	3.285	3.872	3.804	5.330	1.373	2.894	3.396	3.339
		(0.419)	(1.197)	(1.286)	(1.280)		(0.342)	(0.914)	(0.981)	(0.969)		(0.310)	(0.791)	(0.846)	(0.835)
	$\tau = 25$					$\tau = 50$					$\tau = 100$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	2.042	1.228	2.809	3.467	3.528	1.736	1.145	2.334	2.829	2.875	1.464	1.092	1.930	2.287	2.320
		(0.269)	(0.690)	(0.865)	(0.880)		(0.246)	(0.555)	(0.676)	(0.683)		(0.229)	(0.439)	(0.517)	(0.516)
Nik	4.760	1.250	2.569	3.142	3.134	3.941	1.161	2.153	2.600	2.592	3.220	1.106	1.802	2.137	2.130
		(0.280)	(0.631)	(0.778)	(0.775)		(0.255)	(0.513)	(0.619)	(0.619)		(0.236)	(0.412)	(0.482)	(0.481)
UK	4.348	1.185	2.710	3.439	3.454	3.575	1.112	2.266	2.816	2.825	2.871	1.065	1.880	2.274	2.280
		(0.279)	(0.678)	(0.906)	(0.914)		(0.258)	(0.544)	(0.701)	(0.708)		(0.241)	(0.427)	(0.527)	(0.533)
AU	5.035	1.205	2.779	3.421	3.445	4.130	1.123	2.311	2.790	2.811	3.281	1.071	1.911	2.251	2.269
		(0.265)	(0.689)	(0.900)	(0.870)		(0.242)	(0.553)	(0.697)	(0.673)		(0.224)	(0.437)	(0.525)	(0.508)
*TB*1	4.580	1.236	2.495	2.968	2.937	3.514	1.148	2.090	2.454	2.426	2.649	1.094	1.749	2.017	1.994
		(0.280)	(0.592)	(0.712)	(0.704)		(0.255)	(0.479)	(0.562)	(0.559)		(0.234)	(0.381)	(0.433)	(0.432)
*TB*2	4.129	1.222	2.362	2.742	2.701	3.250	1.137	1.996	2.290	2.259	2.502	1.086	1.686	1.906	1.880
		(0.270)	(0.622)	(0.661)	(0.654)		(0.245)	(0.504)	(0.529)	(0.528)		(0.224)	(0.400)	(0.413)	(0.417)

Note: *Emp* stands for the empirical Lo's statistic, $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the mean and standard deviation of Lo's statistics based on the corresponding 1000 simulated time series with pertinent k (Lognormal model).

Table 4: Number of rejections for Lo's R/S statistic.

	$\tau = 0$								$\tau = 5$								$\tau = 10$							
	$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$	
	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡
Dow	324	154	999	998	1000	1000	1000	1000	121	47	996	986	1000	999	1000	1000	66	16	986	955	999	991	998	992
Nik	444	250	999	996	1000	998	1000	1000	172	67	990	971	996	991	998	995	86	30	970	926	989	978	995	979
UK	191	108	999	994	999	998	1000	1000	106	37	985	971	997	992	1000	997	60	15	973	943	992	981	997	988
AU	276	137	1000	999	1000	1000	1000	1000	114	36	998	984	998	993	1000	995	61	14	984	952	993	985	996	993
TB1	423	249	1000	999	1000	1000	999	999	162	64	996	972	998	992	995	987	82	26	969	909	991	970	986	964
TB2	419	227	997	987	1000	998	1000	999	145	62	965	916	998	978	995	971	75	19	919	828	977	946	970	936
	$\tau = 25$								$\tau = 50$								$\tau = 100$							
	$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$		$k = 5$		$k = 10$		$k = 15$		$k = 20$	
	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡	†	‡
Dow	18	2	923	852	983	943	981	960	7	1	795	640	919	845	934	870	1	1	542	341	784	628	793	653
Nik	27	4	864	754	961	915	958	917	8	1	688	506	892	766	880	755	1	1	415	250	689	518	681	512
UK	16	4	907	802	971	937	977	936	8	1	752	586	905	834	915	840	4	1	496	300	760	618	764	602
AU	15	3	918	835	967	933	984	959	5	1	771	616	902	823	928	833	1	1	517	335	754	606	762	612
TB1	22	3	840	706	932	874	935	874	5	1	649	473	833	723	831	722	1	1	380	192	645	436	648	438
TB2	17	3	759	627	912	814	904	806	4	1	956	397	796	620	758	597	2	1	319	166	524	335	506	312

Note: $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the number of rejections at 95% (†) and 99% (‡) confidence levels (these intervals are given by [0.809, 1.862] and [0.721, 2.098], respectively) for the 1000 simulated time series.

Table 5: Lo's modified R/S Hurst exponent H values for the empirical and simulated data.

	$\tau = 0$					$\tau = 5$					$\tau = 10$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	0.620	0.556	**0.671**	**0.699**	**0.702**	0.607	0.540	**0.647**	**0.672**	**0.674**	0.597	0.532	**0.633**	**0.658**	**0.660**
		(0.024)	(0.030)	(0.032)	(0.032)		(0.024)	(0.029)	(0.031)	(0.030)		(0.024)	(0.028)	(0.030)	(0.030)
Nik	0.723	0.564	0.664	**0.688**	**0.688**	0.705	0.544	0.637	**0.660**	**0.660**	0.693	0.534	0.623	**0.646**	**0.646**
		(0.025)	(0.029)	(0.031)	(0.030)		(0.025)	(0.029)	(0.030)	(0.029)		(0.025)	(0.028)	(0.029)	(0.029)
UK	0.712	0.547	**0.662**	**0.693**	**0.693**	0.696	0.533	0.641	**0.670**	**0.671**	0.685	0.526	0.629	**0.657**	**0.657**
		(0.026)	(0.031)	(0.035)	(0.035)		(0.026)	(0.030)	(0.033)	(0.033)		(0.026)	(0.029)	(0.032)	(0.032)
AU	0.726	0.554	0.670	**0.697**	**0.699**	0.711	0.538	0.646	**0.672**	**0.673**	0.700	0.530	0.633	**0.657**	**0.658**
		(0.025)	(0.030)	(0.034)	(0.033)		(0.025)	(0.029)	(0.033)	(0.031)		(0.024)	(0.029)	(0.032)	(0.031)
TB1	0.746	0.564	0.666	**0.687**	**0.687**	0.721	0.544	0.638	**0.659**	**0.658**	0.704	0.534	0.623	**0.644**	**0.643**
		(0.027)	(0.029)	(0.031)	(0.031)		(0.027)	(0.029)	(0.030)	(0.030)		(0.026)	(0.028)	(0.030)	(0.029)
TB2	0.724	0.564	0.657	**0.677**	**0.675**	0.704	0.542	0.630	**0.649**	**0.647**	0.689	0.533	0.616	**0.634**	**0.632**
		(0.026)	(0.033)	(0.030)	(0.031)		(0.026)	(0.032)	(0.030)	(0.030)		(0.025)	(0.031)	(0.029)	(0.029)

	$\tau = 25$					$\tau = 50$					$\tau = 100$				
	Emp	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$	*Emp*	$k = 5$	$k = 10$	$k = 15$	$k = 20$
Dow	0.578	0.520	**0.610**	**0.632**	**0.634**	0.560	0.512	**0.590**	**0.610**	**0.612**	0.542	0.507	**0.569**	**0.587**	**0.589**
		(0.024)	(0.028)	(0.029)	(0.028)		(0.023)	(0.027)	(0.027)	(0.027)		(0.023)	(0.026)	(0.026)	(0.025)
Nik	0.671	0.522	0.600	**0.622**	**0.621**	0.650	0.514	0.581	**0.601**	**0.602**	0.628	0.509	0.561	**0.580**	**0.580**
		(0.025)	(0.028)	(0.028)	(0.028)		(0.024)	(0.027)	(0.027)	(0.027)		(0.023)	(0.026)	(0.026)	(0.026)
UK	0.662	0.516	0.607	**0.632**	**0.633**	0.641	0.509	0.587	**0.611**	**0.611**	0.617	0.504	0.567	**0.588**	**0.588**
		(0.025)	(0.028)	(0.031)	(0.030)		(0.025)	(0.027)	(0.029)	(0.029)		(0.024)	(0.026)	(0.027)	(0.027)
AU	0.679	0.518	0.610	**0.632**	**0.633**	0.657	0.510	0.589	**0.610**	**0.611**	0.631	0.505	0.569	**0.586**	**0.588**
		(0.024)	(0.028)	(0.031)	(0.029)		(0.024)	(0.027)	(0.029)	(0.028)		(0.023)	(0.026)	(0.027)	(0.026)
TB1	0.672	0.521	0.600	**0.619**	**0.618**	0.642	0.513	0.580	**0.598**	**0.597**	0.610	0.507	0.560	**0.576**	**0.575**
		(0.026)	(0.027)	(0.029)	(0.028)		(0.025)	(0.026)	(0.027)	(0.027)		(0.024)	(0.025)	(0.026)	(0.026)
TB2	0.661	0.520	0.593	**0.610**	**0.609**	0.633	0.512	0.574	**0.590**	**0.589**	0.604	0.507	**0.556**	**0.570**	**0.568**
		(0.025)	(0.030)	(0.028)	(0.028)		(0.024)	(0.029)	(0.027)	(0.027)		(0.023)	(0.027)	(0.025)	(0.026)

Note: *Emp* stands for the empirical value of Lo's Hurst exponent, $k = 5$, $k = 10$, $k = 15$ and $k = 20$ refer to the mean and standard deviation of Lo's Hurst exponent based on the corresponding 1000 simulated time series with different k. Boldface numbers are those cases in which empirical Hs fall into the corresponding 2.5 to 97.5 percent quantiles of the 1000 simulation-based values of H.

4 Summary and concluding remarks

In this paper, we have investigated the scaling behaviour of estimated Markov-switching multifractal models with Lognormal volatility components. Based on the empirical estimates via GMM, we have studied the simulated time series and compared their autocorrelation functions with the ones from empirical data. In addition to these qualitative comparisons, we have also calculated the empirical and simulated scaling exponents by using the Generalized Hurst exponent and the modified R/S approaches. Comparing the results from the Lognormal model to our previous study on the Binomial model [7], we observe that there is not much difference between these discrete and continuous versions of multifractal processes. This finding is also in line with the very similar goodness-of-fit and forecasting performance of MSM models reported in [10]. Our results also demonstrate that the MSM models with a relatively large number of volatility components ($k \geq 10$) are required to capture the long-term dependence of absolute values of returns.

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