A Hierarchical Network Model as Birth and Death Process in Random Environment *

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Abstract

In this paper we look at a hierarchical network model for a herding mechanism. We introduce a new class of agents in the network. These agents are not necessary trading on the market, but are connected to all other agents in the network and influence the opinions about the future of the market. In mathematical terms, we replace the birth and death processes in the herding mechanism by birth and death processes in random environment. In this article we show that these birth and death processes in random environment converge to a switching diffusion process if the number of agents on the market grows to infinity.

1 Introduction

Starting with the work [11] of Kirman, herding models for financial markets behaviour have been investigated in a various papers, see e.g. [1], [2], [3], [4], [5], [8], [12]. In these models, the herding is mathematically formulated as a birth and death process. In [5] it is shown that the random network - where any agent is randomly connected to a fixed proportion of all the other agents - is the prototypical network structure that leads to the herding model [3], [4]. They show that it is also important that the number of connections grows linear with the number of agents on the market. In our model we start with a underlying fully connected model, see Figure 1, where all agents are

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connected. In our extension of the model, we include a further cluster of agents that influences all other agents, see Figure 2. The term agent is used here in a very broad sense for any mechanism which influences the behaviour of the trading agents, like information sources or macroeconomic data.

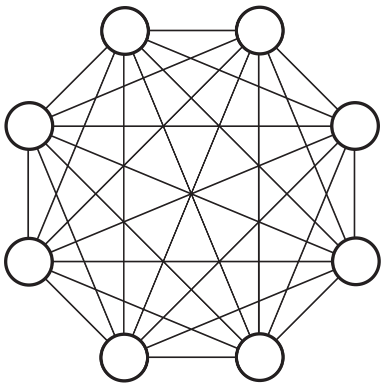

Figure 1: A fully connected network

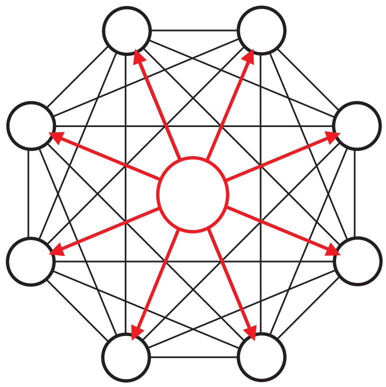

Figure 2: The network of our new model

In Section 2, we describe the new model mathematically and show in the following sections that these birth and death processes in random environment converge to a switching diffusion process if the number of agents grows to infinity. To derive this, we will first take a look at the semigroups and the generators of the processes Y^N in Section 3. After that, we see in Section 4 that the generators of the birth and death processes in random environment converge to a generator of a switching diffusion process. An important step is to calculate that the generator of a switching diffusion process generates a Feller semigroup. We do this in Section 5. Finally we use all these considerations to obtain in Section 6 that the birth and death processes in random environment converge weakly to a switching diffusion process.

2 The model

In this section we give a mathematical definition of the enlarged herding mechanism. In our new model we add a new agent to the network. We call him the super agent. This agent has a finite set of opinions about the future market behaviour. We model the opinion and the switching of the opinions of the super agent by a continuous Markov jump process on a finite set

$\mathcal{M} = \{1, \ldots, M\}$. The state of this process at time t represents the opinion of the super agent at this time. A jump of the process stands for a switching of the opinion. The herding mechanism of the other agents is modelled, like in the cited models, by a birth and death process. This process stands for the number of optimistic agents. The difference to the cited models is that the birth and death rates are now not only functions of the number of optimistic agents but also depend on the opinion of the super agent. This is stated formally in the following way

Let $(Z_t)_{t\geq 0}$, describing the super agent, be a homogeneous Markov process on $\mathcal{M}$ with transition probabilities

$$\begin{aligned} P(Z_{t+\delta} = \bar{z} | Z_t = z) &= q(z, \bar{z})\delta + o(\delta), \;\; \bar{z} \neq z, \\ P(Z_{t+\delta} = z | Z_t = z) &= 1 + q(z, z)\delta + o(\delta). \end{aligned}$$

For all $N \in \mathbb{N}$ the herding of the other agents $(\tilde{X}_t^N)_{t\geq 0}$ is a process on $\{0, \ldots, N\}$ with transition probabilities

$$P(\tilde{X}_{t+\delta}^N = \bar{n} | \tilde{X}_t^N = n, Z_t = z) = \pi_N(n, \bar{n}, z)\delta + o(\delta)$$

with

$$\begin{aligned} \tilde{\pi}_N(n, n+1, z) &= (N - n)\left(a_1(z) + bn\right), \\ \tilde{\pi}_N(n, n-1, z) &= n(a_2(z) + b(N - n)), \\ \tilde{\pi}_N(n, \bar{n}, z) &= 0 \qquad \tilde{n} \notin \{n, n+1, n-1\}. \end{aligned}$$

Then $(\bar{X}_t^N, Z_t)_{t\geq 0}$ is a birth and death process in random environment Z. To show the convergence to a diffusion process we transform the process $\tilde{X}^N$. We will look at the transformed process $X_t^N = \frac{2\tilde{X}_t^N}{N} - 1$ on

$$E_N := \left\{-1, -1 + \frac{2}{N}, \ldots, 1 - \frac{2}{N}, 1\right\}.$$

$(X_t^N, Z_t)_{t\geq 0}$ is a birth and death process in random environment with birth and death rates

$$\begin{aligned} \lambda_N(x, z) &= \frac{N^2}{4}(1 - x)\left(\frac{2a_1(z)}{N} + b(1 + x)\right), \\ \mu_N(x, z) &= \frac{N^2}{4}(1 + x)\left(\frac{2a_2(z)}{N} + b(1 - x)\right), \\ \pi_N(x, \bar{x}, z) &= 0 \qquad \bar{x} \notin \{x, x + \frac{2}{N}, x - \frac{2}{N}\}. \end{aligned}$$

Let $Y^N = (X^N, Z)$.

3 Semigroup and generator of Y^N

In this section we calculate the semigroup and the generator of the birth and death processes.

For the semigroup T^N, $T^N(t) : B(E_N \times \mathcal{M}) \to B(E_N \times \mathcal{M})$ generated by Y^N we obtain for any $f \in B(E_N \times \mathcal{M})$

$$\begin{aligned}
T^N(t)f(x,z) - f(x,z) &= E\left[f(X_t, Z_t) - f(x,z) | X_0 = x, Z_0 = z\right] \\
&= (\lambda_N(x,z)t + o(t))\left[f(x + \frac{2}{N}, z) - f(x,z)\right] \\
&+ (\mu_N(x,z)t + o(t))\left[f(x + \frac{2}{N}, z) - f(x,z)\right] \\
&+ \sum_{\bar{z} \neq z} (q(z,\bar{z})t + o(t))\left[f(x,\bar{z}) - f(x,z)\right].
\end{aligned}$$

Hence the generator of Y_N is

$$\begin{aligned}
A_N f(x,z) &= \lim_{t\to 0} \frac{T_N(t)f(x,z) - f(x,z)}{t} \\
&= \lim_{t\to 0}\left(\lambda_N(x,z) + \frac{o(t)}{t}\right)\left[f(x + \frac{2}{N}, z) - f(x,z)\right] \\
&+ \lim_{t\to 0}\left(\mu_N(x,z) + \frac{o(t)}{t}\right)\left[f(x - \frac{2}{N}, z) - f(x,z)\right] \\
&+ \lim_{t\to 0}\sum_{\bar{z}\neq z}\left(q(z,\bar{z}) + \frac{o(t)}{t}\right)\left[f(x,\bar{z}) - f(x,z)\right] \\
&= \lambda_N(x,z)\left[f(x + \frac{2}{N}, z) - f(x,z)\right] \\
&+ \mu_N(x,z)\left[f(x - \frac{2}{N}, z) - f(x,z)\right] \\
&+ \sum_{\bar{z}\neq z} q(z,\bar{z})\left[f(x,\bar{z}) - f(x,z)\right].
\end{aligned}$$

Using

$$Qf(x,z) = \sum_{\bar{z}\neq z} q(z,\bar{z})\left[f(x,\bar{z}) - f(x,z)\right] = \sum_{\bar{z}\in\mathcal{M}} q(z,\bar{z})f(x,\bar{z}) \qquad (1)$$

we get

$$
\begin{aligned}
A_N f(x,z) &= \lambda_N(x,z)\left[f(x+\frac{2}{N},z) - f(x,z)\right] \\
&+ \mu_N(x,z)\left[f(x-\frac{2}{N},z) - f(x,z)\right] + Qf(x,z).
\end{aligned}
$$

4 Convergence of the generator

In this part we show that the generators from Section 3 converge to the generator of a switching diffusion process if the number of agents tends to infinity. This is done by a Taylor approximation.

Let

$$
\begin{aligned}
B(x,z) &= -\left[(a_1(z)+a_2(z))x + a_2(z) - a_1(z)\right] \\
\sigma^2(x,z) &= b(1-x^2).
\end{aligned}
$$

Theorem 4.1. *Let* $f \in C_0(\Omega) = C_0([-1,1]\times\mathcal{M})$ *and* $f(\cdot,z) \in C^2([-1,1])$ *with continuous extension of the derivatives to* -1 *and* 1 *for all* $z \in \mathcal{M}$. *Let*

$$
Af(x,z) = B(x,z)\frac{df(x,z)}{dx} + \sigma^2(x,z)\frac{d^2 f(x,z)}{dx^2} + Qf(x,z).
$$

Then

$$
A_N f \to Af \ \textit{uniformly.}
$$

Proof. For all $(x,z) \in E_N \times \mathcal{M}$ we obtain

$$
\begin{aligned}
A_N f(x,z) &= \lambda_N(x,z)\left[f(x,z) + \frac{2}{N}\frac{df(x,z)}{dx} + \frac{2}{N^2}\frac{d^2 f(x,z)}{dx^2} + o(\frac{1}{N^2}) - f(x,z)\right] \\
&+ \mu_N(x,z)\left[f(x,z) - \frac{2}{N}\frac{df(x,z)}{dx} + \frac{2}{N^2}\frac{d^2 f(x,z)}{dx^2} + o(\frac{1}{N^2}) - f(x,z)\right] \\
&+ Qf(x,z) \\
&= \frac{2}{N}(\lambda_N(x,z) - \mu_N(x,z))\frac{df(x,z)}{dx} + \frac{2}{N^2}(\lambda_N(x,z)+\mu_N(x,z))\frac{d^2 f(x,z)}{dx^2} \\
&+ (\lambda_N(x,z)+\mu_N(x,z))o(\frac{1}{N^2}) + Qf(x,t).
\end{aligned}
$$

We have

$$
\lambda_N(x,z) - \mu_N(x,z) = \frac{N}{2}B(x,z)
$$

and

$$\lambda_N(x,z) + \mu_N(x,z) = \frac{N^2}{2}\sigma^2(x,z) + o(\frac{1}{N^2}).$$

Furthermore

$$\begin{aligned} A_N f(x,z) &= -\frac{2}{N}\frac{N}{2}B(x,z)\frac{df(x,z)}{dx} + \frac{2}{N^2}\left(\frac{N^2}{2}\sigma^2(x,z))\right)\frac{d^2f(x,z)}{dx^2} \\ &+ o(1) + Qf(x,z) \\ &= Af(x,z) + o(1). \end{aligned}$$

Obviously, this convergence is uniform in (x, z). □

5 The semigroup of the limiting generator

In this section we show that the generator A with

$$Af(x,z) = B(x,z)\frac{df(x,z)}{dx} + \sigma^2(x,z)\frac{d^2f(x,z)}{dx^2} + Qf(x,z) \tag{2}$$

generates a Feller semigroup on $C_0(\Omega)$.

Theorem 5.1. *The generator A defined by* (2) *generates a Feller semigroup $(T_t)_{t\geq 0}$ on $C_0(\Omega)$.*

Before we prove this result, we recapitulate some known results. The first is the important Theorem of Hille and Yosida, that characterizes the generator of Feller semigroups.

Theorem 5.2 (Hille and Yosida). *Let A be a linear operator on $C_0(\Omega)$ with domain $\mathcal{D}$. Than A is closable and its closure $\bar{A}$ is the generator of a Feller semigroup on $C_0(\Omega)$ iff these conditions hold:*

1. *$\mathcal{D}$ is dense in $C_0(\Omega)$;*
2. *the range of $\lambda_0 - A$ is dense in $C_0(\Omega)$ for some $\lambda_0 > 0$;*
3. *if $f^+ \leq f(x)$ for some $f \in \mathcal{D}$ and $x \in \Omega$, then $Af(x) \leq 0$.*

Proof. See [10] Theorem 19.11, p.375.
The third condition is known as the positive maximum principle. For the second result we need two further definitions.

Definition 5.1 (dissipative)**.** A linear operator $A : \mathcal{D}(A) \to X, \mathcal{D}(A) \subseteq X$, is called dissipative, if

$$\|\lambda u - Au\| \geq \lambda \|u\|$$

holds for all $\lambda > 0$ and $u \in \mathcal{D}(A)$.

Lemma 5.1. *A linear operator A on C_0 satisfying the positive maximum principle is dissipative.*

Proof. See [10], p.377.

Definition 5.2 (A-bounded)**.** Let $(A, \mathcal{D}(A))$ and $(Q, \mathcal{D}(Q))$ be two linear operators on X such that $\mathcal{D}(A) \subseteq \mathcal{D}(Q)$ and for some $\alpha \in [0, 1)$ and $\beta \geq 0$

$$\|Qu\| \leq \alpha \|Au\| + \beta \|u\|$$

holds for all $u \in \mathcal{D}(A)$. Then the operator Q is $A-$bounded.

With this definitions we can cite a useful result from perturbation theory.

Theorem 5.3. *Let $(A, \mathcal{D}(A))$ be a linear operator on a Banach space $(X, \|\cdot\|_X)$ such that $\mathcal{D}(A)$ is dense in X. Suppose that A is dissipative and the range of $\lambda - A$ is dense in X for some $\lambda > 0$. If $(Q, \mathcal{D}(Q))$ is an $A-$bounded dissipative operator on X, then $(A + Q, \mathcal{D}(A))$ is closable and its closure generates a strongly continuous contraction semigroup.*

Proof. See [9] Theorem 4.4.3, p.320.

Finally we need a result that shows us the connection between positive semigroups and the positivity of the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ of the generator A of the semigroup.

Theorem 5.4. *A strongly continuous semigroup $(T_t)_{t \geq 0}$ on a Banach lattice X is positive if and only if the resolvent $R(\lambda, A)$ of its generator A is positive for all sufficient large λ.*

Proof. See [6] Theorem 1.3, Section 2, Chapter VI.

We can now prove Theorem 5.1.

Proof of Theorem 5.1. First we write $A = \tilde{A} + S$ with

$$\tilde{A}f(x, z) = B(x, z)\frac{df(x, z)}{dx} + \sigma^2(x, z)\frac{d^2 f(x, z)}{dx^2} \text{ and } Sf(x, z) = Qf(x, z).$$

Our first step is to show that the operator $\tilde{A}$ generates a Feller semigroup. We do this with the Theorem 5.2 of Hille and Yosida. Define for all $z \in \mathcal{M}$ $A^z : C_0[-1, 1] \to C_0[-1, 1]$ with $A^z f = \tilde{A}f(\cdot, z)$. Then it is easy to see that the conditions one and three from Theorem 5.2 hold for the generator $\tilde{A}$ with $\mathcal{D} = \{f \in C_0([-1, 1] \times \mathcal{M}) : f(\cdot, z) \in \mathcal{D}(A^z) \text{ for all } z \in \mathcal{M}\}$. For the second condition we have to show that $(\lambda - \tilde{A})\mathcal{D}$ is dense in $C_0([-1, 1] \times \mathcal{M})$. Let $f \in C_0([-1, 1] \times \mathcal{M})$ and $\varepsilon > 0$. Define $f^z(x, \tilde{z}) = f(x, \tilde{z})\delta_z(\tilde{z})$. We know that A^z is a generator of a Feller semigroup and we know from Theorem 5.2 that $(\lambda - A^z)\mathcal{D}(A^z)$ is dense in $C_0([-1,1])$. Then there exists a $f_\varepsilon^z \in \mathcal{D}(A^z)$ with

$$\|(\lambda - A^z)f_\varepsilon^z - f(\cdot, z)\|_{C_0([-1,1])} \leq \frac{\varepsilon}{M}.$$

From this it follows with $\tilde{f}_\varepsilon^z : C_0([-1,1] \times \mathcal{M})$, $\tilde{f}_\varepsilon^z(x, \tilde{z}) = f_\varepsilon^z(x)\delta_z(\tilde{z})$

$$\|(\lambda - \tilde{A})\tilde{f}_\varepsilon^z - f^z\| \leq \frac{\varepsilon}{M}.$$

Hence

$$\begin{aligned} \|(\lambda - \tilde{A}) \sum_{z \in \mathcal{M}} \tilde{f}_\varepsilon^z - f\| &= \| \sum_{z \in \mathcal{M}} (\lambda - \tilde{A})\tilde{f}_\varepsilon^z - \sum_{z \in \mathcal{M}} f^z\| \\ &\leq \sum_{z \in \mathcal{M}} \|(\lambda - \tilde{A})\tilde{f}_\varepsilon^z - f^z\| \\ &\leq \varepsilon. \end{aligned}$$

Furthermore $\sum_{z\in\mathcal{M}} \tilde{f}_\varepsilon^z \in \mathcal{D}$ and $(\lambda - \tilde{A})\mathcal{D}$ is dense in $C_0([-1,1] \times \mathcal{M})$. Now Theorem 5.2 shows that $\tilde{A}$ generates a Feller semigroup.
Let us take a look at the operator S. We see that S is a bounded operator. Furthermore, with (1), it is easy to see that S fulfills the positive maximum principle and hence, with Lemma 5.1, S is dissipative. So with Theorem 5.3, we get that $A = \tilde{A} + S$ is the generator of a strongly continuous contraction semigroup on $C_0(\Omega)$. Now we have to show that A generates a positive semigroup. From Theorem 5.4, we know that this follows if $R(\lambda, A)$ is positive for all $\lambda > 0$. Let $g \in C_0(\Omega)$ a positive function and put $f = R(\lambda, A)g$, so that $g = (\lambda - A)f$. It is easy to check that the generator A fulfills the positive maximum principle for $\mathcal{D} = \mathcal{D}(A)$. If $\inf_{x\in\Omega} f(x) < 0$, we choose some $x_0 \in \Omega$ with $f(x_0) \leq \min\{f, 0\}$. With the positive maximum principle we get $Af(x_0) \geq 0$, and so

$$\inf_{x\in\Omega} g(x) = \inf_{x\in\Omega} (\lambda - A)f(x) \leq (\lambda - A)f(x_0) \leq \lambda f(x_0) = \lambda \inf_{x\in\Omega} f(x) < 0.$$

But g is a positive function and so $\inf_{x\in\Omega} f(x) \geq 0$. □

6 Convergence to a switching diffusion

In this section we show that the semigroups T^N converge to the semigroup T and that there exists a switching diffusion process X corresponding to A, such that X^N converge to X in $D_\Omega[0, \infty)$.
To show this, we need two textbook results.

Theorem 6.1. *For $N \in \mathbb{N}$ let L_N and L be Banach spaces and $\pi_N : L \to L_N$ a bounded linear transformation. Let T^N and T be strongly continuous contraction semigroups on L_N and L with generators A^N and A. Let $\mathcal{D}$ be a core for A. Then the following are equivalent:*

1. *For each $f \in L$, $T_t^N \pi_N f \to T_t f$ for all $t \geq 0$, uniformly on bounded intervals.*

2. *For each $f \in L$, $T_t^N \pi_N f \to T_t f$ for all $t \geq 0$.*

3. *For each $f \in \mathcal{D}$, there exists $f_N \in \mathcal{D}(A^N)$ for each $n \geq 1$ such that $f_N \to f$ and $A_N f_N \to Af$.*

Proof. See [7] Theorem 6.1, Chapter 1.

Theorem 6.2. *Let E, E_N $N \in \mathbb{N}$ be metric spaces with E locally compact and separable. For $N \in \mathbb{N}$ let $\eta_N : E_N \to E$ be measurable, let T^N be a semigroup on $B(E_N)$ given by a transition function, and suppose Y^N is a Markov process in E_N corresponding to T^N such that $X^N = \eta_N \circ Y^N$ has sample paths in $D_E[0,\infty)$. Define $\pi_N : B(E) \to B(E_N)$ by $\pi_N f = f \circ \eta_N$. Suppose that T is a Feller semigroup on $C_0(E)$ and that for each $f \in C_0(E)$ and $t \geq 0$, $T_t^N \pi f \to T_t f$. If X_0^N has a limiting distribution ν, then there is a Markov process X corresponding to T with initial distribution ν and sample paths in $D_E[0,\infty)$, and X^N converge to X in $D_E[0,\infty)$.*

Proof. See [7] Theorem 2.11, Chapter 4.

We now use these results with $E = [-1,1] \times \mathcal{M}$,

$$E_N = \{-1, -1 + \frac{2}{N}, \dots, 1 - \frac{2}{N}, 1\} \times \mathcal{M}, L = C_0(E), L_N = C_0(E_N)$$

and $\eta_N = id, \pi_N f = f \circ \eta_N$. The third statement from Theorem 6.1 follows from Theorem 4.1 and so we get

$$T_t^N \pi_N f \to T_t f \text{ for all } f \in C_0(\Omega).$$

Then we can use Theorem 6.2 to obtain a switching diffusion process X with semigroup T and generator A such that

$$X^N \to X \text{ in } D_\Omega[0,\infty).$$

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