# Diffusion Approximation of Birth and Death Processes with Applications in Financial Market Herding Models*

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**Abstract**

We take a mathematical look on the herding mechanism in the herding model for financial markets as described in [4]. The main aspect of this article is the convergence of the birth and death process of optimistic agents to a diffusion process. Apart from this we look on some aspects of the limiting diffusion process with concentration on the boundary point behaviour.

## 1 Introduction

Starting with the work [10] of Kirman in the year 1993, herding models for financial markets are investigated in a variety of papers, see [1], [2], [3], [4], [7], [11]. In this article we concentrate on the model described in [4]. They consider a financial market with a fixed number of agents $N$. These agents are either optimistic or pessimistic about the future market behaviour. In this paper we take a mathematical look on the process of optimistic agents for which the standard mathematical model is modelled as a birth and death process. A death stands for an optimistic agent who chances his opinion and a birth stands for an pessimistic agent who gets optimistic. This swing of opinion can happen due to two factors. The first is that the agent chances his opinion without influence from his peers. For a change due to group pressure

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it is important to assume that the agents are connected in a network and communicate with each other. Possible underlying network structures are considered in [5]. We refer to the cited papers for detailed information on the economic background.
First we give a mathematical model of the birth and death process in Section 2. Then we show in Section 3 that this birth and death process converges to a diffusion process if the number of agents on the market tends to infinity. After that we take a look at the boundary behaviour of the limiting diffusion process in Section 4.

# 2 The model

For all $N \in \mathbb{N}$ let $(Z_t^N)_{t\in[0,\infty)}$ be a homogeneous Markov process on $\{0, ..., N\}$ with transition probabilities

$$P(Z_{t+\delta}^N = \tilde{n} | Z_t^N = n) = \pi(n, \tilde{n})\delta + o(\delta)$$

with

$$\begin{aligned} \pi_N(n, n+1) &= (N-n)\,(a+bn)\,,\ n = 0, \ldots N-1 \\ \pi_N(n, n-1) &= n(a + b(N-n)),\ n = 1, \ldots, N \\ \pi_N(n, \tilde{n}) &= 0 \qquad \tilde{n} \notin \{n, n+1, n-1\}. \end{aligned}$$

Then $(Z_t^N)_t$ is a birth and death process with birth rate $\lambda_N(n) = \pi_N(n, n+1)$ and death rate $\mu_N(n) = \pi_N(n, n-1), n \in \{0, ..., N\}$ with $\lambda_N(N) = \mu_N(0) = 0$.

# 3 The approximation

In the following section we look at the limit of the processes $(Z_t^N)_t$ for $N \to \infty$. This is the case when the number of trading agents on the market tends to infinity, and the limit may be seen to provide a good approximation for a market with a large number of agents.
To show this, we look at the following transformation

$$X_t^N = \frac{2Z_t^N}{N} - 1.$$

Then $X^N = (X_t^N)_{t\geq 0}$ is a homogeneous Markov process on

$$E_N := \left\{-1, -1 + \frac{2}{N}, \ldots, 1 - \frac{2}{N}, 1\right\}$$

with birth and death rates

$$\lambda_N(x) = \pi_N(x, x + \frac{2}{N}) \;=\; \frac{N^2}{4}(1-x)\left(\frac{2a}{N} + b(1+x)\right), \tag{1}$$

$$\mu_N(x) = \pi_N(x, x - \frac{2}{N}) \;=\; \frac{N^2}{4}(1+x)\left(\frac{2a}{N} + b(1-x)\right), \tag{2}$$

for $x \in [-1, 1]$. We will show that the processes $X^N$ converges to a diffusion process on $[-1, 1]$. For this we need various results from the theory of Markov processes and we refer to [6] and [8] as excellent sources.

## 3.1 General definitions

Let $L$ be a Banach space.

**Definition 3.1** (semigroup)**.** A family $(T(t))_{t\geq 0}$ of bounded linear operators $T(t) : L \to L$ is called a *semigroup* if

- $T(0) = Id$ and
- $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$.

A semigroup $(T(t))_t$ is called *strongly continuous* if

$$\lim_{t\to 0} T(t)f = f$$

for all $f \in L$.
A semigroup $(T(t))_t$ is called a *contraction* semigroup if $\|T(t)\| \leq 1$ holds for all $t \geq 0$.

**Definition 3.2** (generator)**.** Let $(T(t))_t$ be a semigroup. The *generator* $A$ of this semigroup is defined by

$$Af = \lim_{t\to 0} \frac{T(t)f - f}{t}$$

for all $f \in \mathfrak{D}(A)$. $\mathfrak{D}(A)$ is the subspace of $L$ where this limit exists.

**Definition 3.3.** Let $X = (X_t)_t$ be a Markov process. Then the semigroup $(T(t))_t$ with

$$T(t)f(x) = E(f(X_t)|X_0 = x)$$

is called the semigroup of $X$.

## 3.2 Semigroups and generators of the processes $(X_t^N)_t$

### 3.2.1 The semigroups

The semigroups of the birth and death processes $(X_t^N)_t$ are given by $(T_N(t))_t : B(E_N) \to B(E_N)$ with

$$(T_N(t)f)(x) = E(f(X_t^N)|X_0^N = x).$$

for all $f \in B(E_N)$ and $x \in E_N$. Here $B(E_N)$ denotes the set of bounded functions from $E_n$ to $\mathbb{R}$.
We have

- $(T_N(0)f)(x) = E(f(X_0^N)|X_0^N = x) = f(x)$ and
- $T_N(s+t) = T_N(s)T_N(t)$.
  The last equation follows directly from the Chapman-Kolmogorov equation.

$(T_N(t))_t$ is a contraction semigroup because for all $f \in B(E_N), x \in E_N$

$$|(T_N(t)f)(x)| = |E(f(X_t^N)|X_0^N = x)| \leq \sup_{y \in E_N} |f(y)| = \|f\|.$$

Let $C_N = \max_{x \in E_N \setminus \{-1\}} |f(x) - f(x - \frac{2}{N})|$. Then

$$\begin{aligned} |(T_N(t)f)(x) - f(x)| &= |E[f(X_t^N) - f(x)|X_0^N = x]| \\ &\leq |(\lambda_N(x)t + o(t))[f(x + \frac{2}{N}) - f(x)]| \\ &+ |(\mu_N(x)t + o(t))[f(x - \frac{2}{N}) - f(x)] + o(t)| \\ &\leq \left( \max_{y \in E_N} \lambda_N(y)t + o(t) \right) C_N \\ &+ \left( \max_{y \in E_N} \mu_N(y)t + o(t) \right) C_N \\ &= C_N \cdot t \left( \max_{y \in E_N} \lambda_N(y) + \max_{y \in E_N} \mu_N(y) + \frac{o(t)}{t} \right). \end{aligned}$$

Furthermore

$$\sup_{x \in E_N} |(T_N(t)f)(x) - f(x)| \leq C_N \cdot t \left( \max_{y \in E_N} \lambda_N(y) + \max_{y \in E_N} \mu_N(y) + \frac{o(t)}{t} \right) \stackrel{t \to 0}{\to} 0,$$

where $\mu_N$ and $\lambda_N$ denote the birth and death rates of the birth and death processes $X^N$. So the semigroup $(T_N(t))_t$ is strongly continuous.

### 3.2.2 The generators of the semigroups $(T_N(t))_t$

We have

$$\begin{aligned} E[f(X_t^N) - f(x)|X_0^N = x] &= (\lambda_N(x)t + o(t))[f(x + \frac{2}{N}) - f(x)] \\ &+ (\mu_N(x)t + o(t))[f(x - \frac{2}{N}) - f(x)] + o(t). \end{aligned}$$

From this

$$\begin{aligned} A_N f(x) &= \lim_{t\downarrow 0} \left(\lambda_N(x) + \frac{o(t)}{t}\right)[f(x + \frac{2}{N}) - f(x)] \\ &+ \lim_{t\downarrow 0} \left(\mu_N(x) + \frac{o(t)}{t}\right)[f(x - \frac{2}{N}) - f(x)] \\ &+ \lim_{t\downarrow 0} \frac{o(t)}{t} \\ &= \lambda_N(x)[f(x + \frac{2}{N}) - f(x)] + \mu_N(x)[f(x - \frac{2}{N}) - f(x)]. \end{aligned}$$

## 3.3 Convergence of the processes $X^N$

In this section we show that the birth and death processes converge to a diffusion process if the number of agents tends to infinity. For this we first show in Theorem 3.1 that the generators $A_N$ of the birth and death processes converge to the generator $A$ of a diffusion process. Then we show in Theorems 3.2-3.4, that there exists a diffusion process $X$ with this generator $A$ and that the birth and death processes $X^N$ converge in $D_{[-1,1]}[0,\infty)$ to this diffusion process $X$; see [6] for a detailed discussion of this convergence.
Let $E = [-1, 1]$. Let

$$\mathcal{D} = \{f \in C^2(E) : f', f'' \text{ have cont. ext. to } -1, 1, \text{ with } f'(-1) = f'(1) = 0\}$$

and set

$$Af(x) = -2axf'(x) + b(1 - x^2)f''(x).$$

**Theorem 3.1.** *For all $f \in \mathcal{D}$*

$$\lim_{N\to\infty} \sup_{y\in E_N} |A_N f(y) - Af(y)| = 0.$$

*Proof.* Let $f \in \mathcal{D}$ and $x \in E_N, x \neq -1, 1$. Then we get by Taylor approximation

$$\begin{aligned} A_N f(x) &= \lambda_N(x)\left[f(x) + \frac{2}{N}f'(x) + \frac{2}{N^2}f''(x) + o(\frac{1}{N^2}) - f(x)\right] \\ &+ \mu_N(x)\left[f(x) - \frac{2}{N}f'(x) + \frac{2}{N^2}f''(x) + o(\frac{1}{N^2}) - f(x)\right] \\ &= \frac{2}{N}(\lambda_N(x) - \mu_N(x))f'(x) + \frac{2}{N^2}(\lambda_N(x) + \mu_N(x))f''(x) \\ &+ (\lambda_N(x) + \mu_N(x))o(\frac{1}{N^2}). \end{aligned}$$

Due to the fact that $f \in \mathcal{D}$, the error term $o(\frac{1}{N^2})$ is uniform in $x$.
Furthermore

$$\lambda_N(x) - \mu_N(x) = \frac{N^2}{4}\frac{-4ax}{N} = -Nax$$

and

$$\lambda_N(x) + \mu_N(x) = \frac{N^2}{4}\left(\frac{4a}{N} + 2b(1-x^2)\right).$$

This shows

$$\begin{aligned} A_N f(x) &= -\frac{2}{N}Naxf'(x) + \frac{2}{N^2}\frac{N^2}{4}\left(\frac{4a}{N} + 2b(1-x^2)\right)f''(x) \\ &+ \frac{N^2}{4}\left(\frac{4a}{N} + 2b(1-x^2)\right)o(\frac{1}{N^2}) \\ &= -2axf'(x) + b(1-x^2)f''(x) \\ &+ \frac{N^2}{4}\left(\frac{4a}{N} + 2b(1-x^2)\right)o(\frac{1}{N^2}) \\ &= Af(x) + \frac{N^2}{4}\left(\frac{4a}{N} + 2b(1-x^2)\right)o(\frac{1}{N^2}). \end{aligned}$$

Hence

$$\begin{aligned} |A_N f(x) - Af(x)| &= |\frac{N^2}{4}\left(\frac{4a}{N} + 2b(1-x^2)\right)o(\frac{1}{N^2})| \\ &\leq \frac{4a}{N}|\frac{N^2}{4}o(\frac{1}{N^2})| + |2b(1-x^2)\frac{N^2}{4}o(\frac{1}{N^2})| \\ &\leq \frac{4a}{N}|\frac{N^2}{4}o(\frac{1}{N^2})| + |2b\frac{N^2}{4}o(\frac{1}{N^2})|. \end{aligned}$$

This shows

$$\sup_{y \in E_N y \neq -1,1} |A_N f(y) - Af(y)| \leq \frac{4a}{N}|\frac{N^2}{4}o(\frac{1}{N^2})| + |2b\frac{N^2}{4}o(\frac{1}{N^2})| \overset{N\to\infty}{\to} 0.$$

For $x = -1$ we have

$$\begin{aligned} |A_N f(-1) - Af(-1)| &\le |A_N f(-1) - A_n f(-1 + \frac{2}{N})| \\ &\quad + A_N f(-1 + \frac{2}{N}) - Af(-1 + \frac{2}{N})| \\ &\quad + |Af(-1 + \frac{2}{N}) - Af(-1)|. \end{aligned}$$

The second term tends to 0 as shown above, the third term by continuity of $Af$ in $-1$. For the first term we have

$$\begin{aligned} \left| A_N f(-1) - A_N f(-1 + \frac{2}{N}) \right| &= \Big| \lambda_N(-1) \left( f(-1 + \frac{2}{N}) - f(-1) \right) \\ &\quad - \lambda_N(-1 + \frac{2}{N}) \left( f(-1 + \frac{4}{N}) - f(-1 + \frac{2}{N}) \right) \\ &\quad + \mu_N(-1 + \frac{2}{N}) \left( f(-1) - f(-1 + \frac{2}{N}) \right) \Big| \\ &= O(N) o(\frac{1}{N}) \to 0, \ N \to \infty, \end{aligned}$$

since $\lambda_N(-1), \lambda_N(-1 + \frac{2}{N}), \mu_N(-1 + \frac{2}{N})$ are of order $N$ and $|f(-1 + \frac{2}{N}) - f(-1)|$, $|f(-1 + \frac{4}{N}) - f(-1 + \frac{2}{N})|$ are of order $o(\frac{1}{N})$ as $f'(-1) = 0$ for $f \in \mathcal{D}$. The result for $x = 1$ follows in the same way. □

This convergence of the generators $A_N$ to the generator $A$. In the following section we show that there exists a process $X$ with generator $A$ so that the birth and death processes $X_N$ converge to $X$ in $D_E[0, \infty)$. We refer to [6] for a discussion of $D_E[0, \infty)$ and the notion of convergence in this space. We need some further theorems and definitions from [6].

**Definition 3.4** (Feller semigroup)**.** A semigroup $(T(t))_t$ on $C(E)$ is called *positive*, if $T(t)$ is a positive operator for all $t \ge 0$. A positive contraction semigroup $(T(t))_t$ on $C(E)$ is called a *Feller semigroup*, if

1. $T(t)C(E) \subseteq C(E)$ for all $t \ge 0$.

2. $T(t)f(x) \stackrel{t \to 0}{\to} f(x)$ for all $f \in C(E), x \in E$.

**Definition 3.5** (core)**.** A subset $D \subseteq \mathfrak{D}(A)$ is called *core* of a generator $A : L \to L$, if the closure of the restriction of $A$ to $D$ is equal to $A$ i.e.,

$$\overline{\{(f, Af) : f \in D\}} = \{(f, Af) : f \in \mathfrak{D}(A)\}.$$

Next, we will show that the generator $A$ generates a Feller semigroup $(T(t))_t$ on $C(E)$.

**Theorem 3.2.** *The generator* $A = \nu(x)\frac{d}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}$ *generates a Feller semigroup* $(T(t))_t$ *on* $C(E)$ *with core* $C^\infty(E)$ *if*

*(i)* $\nu \in C^2(E), \nu \geq 0$ *and* $\nu^{(2)}$ *is bounded and*

*(ii)* $\sigma^2 : E \to \mathbb{R}$ *is Lipschitz continuous.*

*Proof.* See [6], Theorem 2.1, Chapter 8. □

The next result shows that the semigroups of the birth and death processes $T_N(t)$ converge to the semigroup $T$ of the diffusion process.
Define $\eta_N : E_N \to E$ with $\eta_N(x) = x$ and $\pi_N : B(E) \to B(E_N)$ with $\pi_N(f) = f \circ \eta_N$.

**Theorem 3.3.** *Let* $(T_N(t))_t, N \in \mathbb{N}$ *and* $(T(t))_t$ *be strongly continuous contraction semigroups on* $B(E_N)$ *and* $C(E)$ *with generators* $A_N, N \in \mathbb{N}$*, and* $A$*. Let* $D$ *be a core for* $A$*. Then the following are equivalent:*

1. *For each* $f \in C(E)$
$$T_N(t)\pi_N f \to T(t)f$$
*for all* $t \geq 0$*, uniformly on bounded intervals.*

2. *For each* $f \in C(E)$ *and all* $t \geq 0$
$$T_N(t)\pi_N f \to T(t)f.$$

3. *For each* $f \in D$*, there exists* $f_N \in \mathfrak{D}(A_N)$ *for each* $N \geq 1$ *such that* $f_N \to f$ *and* $A_N f_N \to Af$*.*

*Proof.* See [6], Theorem 6.1, Chapter 1. □

**Theorem 3.4.** *Suppose that* $(T(t))_t$ *is a Feller semigroup on* $C(E)$ *and that for each* $t \geq 0$ *and* $f \in C(E)$
$$T_N(t)\pi_N f \to T(t)f.$$
*If* $X_N(0)$ *has limiting distribution* $\nu$*, then there is a Markov process* $X$ *corresponding to* $(T(t))_{t\geq 0}$ *with initial distribution* $\nu$ *and sample paths in* $D_E[0,\infty)$*, and*
$$X_N \to X \text{ in } D_E[0,\infty).$$

*Proof.* See [6], Theorem 2.11, Chapter 4. □

**Conclusion.** To the operator

$$Af(x) = -2axf'(x) + b(1-x^2)f''(x).$$

corresponds a diffusion process $X = (X_t)_t$ on $E = [-1,1]$. In 4.2 we investigate the boundary behaviour of this process at $-1$ and 1. It follows from the results in 4.2 and 8.1.1 of [6] that $\mathcal{D}$ is a core for this process. So it follows from 3.3 and 3.4 that we have the convergence of the normalized processes

$$X^N = \frac{2Z^N}{N} - 1 \to X.$$

# 4 The limiting process

From the form of the generator $A$, we know that the limiting process $X$ is a diffusion process on $[-1,1]$. In this section we calculate the diffusion and drift parameter of the limiting process and derive the boundary behaviour of this process.

## 4.1 Diffusion and drift parameter

**Definition 4.1.** The *diffusion parameter* $\nu$ is given by

$$\nu(x) = \lim_{t\downarrow 0} \frac{E[X_t - X_0|X_0 = x]}{t}$$

and the *drift parameter* $\sigma^2$ is given by

$$\sigma^2(x) = \lim_{t\downarrow 0} \frac{E[(X_t - X_0)^2|X_0 = x]}{t}.$$

With this definition we can calculate the parameters of the limiting diffusion process.

**Theorem 4.1.** *The parameters of the limiting diffusion process $X$ are given by*

$$\nu(x) = -2ax \text{ and } \sigma^2(x) = 2b(1-x)^2 \text{ for all } x \in [-1,1].$$

*Proof.* We calculate the drift parameter from the generator $A$ with the function $f_1(x) = x$. With this function we get

$$\nu(x) = Af_1(x) = -2ax(1) + b(1-x^2)(0) = -2ax.$$

For the calculation of the diffusion parameter we use the functions $f_1$ and $f_2(x) = x^2$. Then we get

$$\begin{aligned}\sigma^2(x) &= Af_2(x) - 2xAf_1(x)\\ &= -2ax(2x) + b(1-x^2)2 - 2x(-2ax)\\ &= 2b(1-x^2).\end{aligned}$$

□

## 4.2 Boundary behaviour of the limiting process

We now provide a discussion of the boundary behaviour of the limiting diffusion and refer to [8] and [9] for a detailed discussion of the basic results as used here.
The boundary behaviour of a diffusion process is characterized by the following two functions.

**Definition 4.2** (scale and speed measure)**.** The *scale function* $S$ is defined by

$$S(x) = \int_0^x s(y)dy \ with \ s(y) = \exp\left\{-\int_0^y \frac{2\nu(z)}{\sigma^2(z)}dz\right\}.$$

From the scale function we obtain the *scale measure* by

$$S[c,d] = S(d) - S(c).$$

The *speed measure* $M$ is defined by

$$M[c,d] = \int_c^d m(x)dx$$

with $m(x) = \frac{1}{\sigma^2(x)s(x)}$.

For the characterization of the boundaries we define the following functions:

1. $u(x) = \int_0^x M[0,y]dS(y)$,
2. $v(x) = \int_0^x S[0,y]dM(y)$.

With these functions a boundary point $r \in \{-1,1\}$ is characterized as

1. regular, if $u(r) < \infty$ and $v(r) < \infty$,
2. entrance, if $u(r) = \infty$ and $v(r) < \infty$,

3. exit, if $u(r) < \infty$ and $v(r) = \infty$.

We now characterize the behaviour of our limiting diffusion process $X$ at the boundaries $r_0 = -1$ and $r_1 = 1$. We will see how this behaviour is influenced by the parameters $a, b$ of our model. Let $\eta = \frac{a}{b} - 1$ and $\delta = -\frac{a}{b}$. Then

1. $u(x) = \frac{1}{b}\int_0^x \int_0^y (1-z^2)^\eta dz (1-y^2)^\delta dy$,
2. $v(x) = \frac{1}{b}\int_0^x \int_0^y (1-z^2)^\delta dz (1-y^2)^\eta dy$.

Based on the symmetry of the functions $u, v$ at the boundaries, we only look at the behaviour at the boundary $r_1 = 1$. We have several cases for different behaviour of the parameters $a$ and $b$.

### 4.2.1 The case $a < b$

In this case, $-1 < \eta, \delta < 0$. It follows

$$\begin{aligned} u(1) &= \frac{1}{b}\int_0^1 \int_0^y (1-z)^\eta (1+z)^\eta dz (1-y)^\delta (1+y)^\delta dy \\ &\leq \frac{1}{b}\int_0^1 \int_0^y (1-z)^\eta dz (1-y)^\delta dy \\ &= \frac{1}{b(\eta+1)}\int_0^1 (1-y)^\delta - (1-y)^{\eta+1+\delta} dy \\ &= \frac{1}{b(\eta+1)}\int_0^1 (1-y)^\delta - 1 dy \\ &= \frac{1}{b(\eta+1)}\left(\frac{1}{b+1} - 1\right) < \infty. \end{aligned}$$

In the same way we show $v(1) < \infty$. Hence our boundaries are regular.

### 4.2.2 The case $a = b$

Then $\eta = 0$ and $\delta = -1$. From this

$$\begin{aligned} u(1) &= \frac{1}{b}\int_0^1 \int_0^y (1-z^2)^\eta dz (1-y^2)^\delta dy \\ &= \frac{1}{b}\int_0^1 y(1-y^2)^{-1} dy \\ &\geq c\int_{\frac{1}{2}}^1 (1-y)^{-1} dy = \infty. \end{aligned}$$

Furthermore

$$\begin{aligned} v(1) &= \frac{1}{b}\int_0^1\int_0^y (1-z^2)^{-1}dzdy \\ &= \frac{1}{b}\int_0^1 arctanh(y)dy \\ &= \frac{1}{b}\ln(2) < \infty. \end{aligned}$$

Therefore the boundaries are entrance boundaries.

### 4.2.3 The case $a > b$

We have $\eta > 0$ and $\delta < -1$, hence

$$\begin{aligned} u(1) &= \frac{1}{b}\int_0^1\int_0^y (1-z^2)^{\eta}dz(1-y^2)^{\delta}dy \\ &\geq \frac{1}{b}\int_0^1 y(1-y^2)^{\eta}(1-y^2)^{\delta}dy \\ &= \frac{1}{b}\int_0^1 y(1-y^2)^{-1}dy = \infty. \end{aligned}$$

For $v$ we obtain

$$\begin{aligned} v(1) &= \frac{1}{b}\int_0^1\int_0^y (1-z)^{\delta}(1+z)^{\delta}dz(1-y)^{\eta}(1+y)^{\eta}dy \\ &\leq \frac{2^{\eta}}{b(\delta+1)}\int_0^1 (1-y)^{\eta} - (1-y)^{\delta+1+\eta}dy \\ &= \frac{2^{\eta}}{b(\delta+1)}\left(\frac{1}{\eta+1} - 1\right) < \infty. \end{aligned}$$

Hence our boundaries are entrance boundaries.

Finally there is one special case, the case $a = 0$. In this case we have no herding behaviour in the model and the switching from one to the other group only takes place by communication with members of the other group. Due to this we expect that the boundaries are exit boundaries in this case. This intuition is right and we can calculate this in the same way as in 4.2.2.

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