Switching Rates and the Asymptotic Behaviour of Herding Models*

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1 Introduction

Continuous-time Markov chains are a popular tool for stochastic modelling in a great variety of fields, ranging from population genetics to communication networks. The mathematical theory of such processes is well understood and provides powerful results for the handling of applied problems. Recently such processes have entered economical theory as agent-based models which are able to explain some of the stylized facts of financial markets. Starting with the work [9] of Kirman, herding model are used to describe the behaviour of agents in financial markets, see [1], [2], [3], [4], [7], [10].
In this note, we shall review some of the relevant mathematical facts and show how they apply to agent-based models and provide insights into the asymptotic behaviour.

2 Basic facts on continuous-time Markov chain

2.1 Generalities

We consider a continuous-time Markov chain $(X_t)_{t\in[0,\infty)}$ with discrete state space S, finite or countably infinite, having right-continuous paths. The Markov properties states that

$$P(X_{t+s} = j|(X_r)_{0\leq r\leq s}, X_s = i) = P(X_{t+s} = j|X_s = i)$$

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for all $t, s \geq 0, i, j \in S$, i.e. the future development, given the past and present, only depends on the present state i and is time-homogeneous. $(P_{ij}(t))_{i,j}$ are stochastic matrices and are called the transition matrices.
The infinitesimal characterics are given by

$$q_{ij} = \lim_{t \downarrow 0} \frac{P_{ij}(t)}{t}, j \neq i, \ q_{ii} = \lim_{t \downarrow 0} \frac{P_{ii}(t) - 1}{t},$$

and we assume $0 < qi = -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ for all states. The q_{ij} are the transition rates and the matrix $Q = (q_{ij})_{i,j}$ is the generator. Q uniquely determines the distribution of the process and we have the matrix differential equation

$$P'(t) = P(t)Q, \text{ resulting in } P(t) = e^{Qt}.$$

A stationary distribution $\pi = (\pi_i)_i$ fulfills the equation $\pi^T Q = 0$. To Q corresponds the infinitesimal generator A which, for any bounded mapping $f : S \to S$, defines a mapping $Af : S \to S$ by

$$Af(i) = \lim_{t \downarrow 0} \frac{\sum_{j \in S} P_{ij}(t)(f(j) - f(i))}{t} = \sum_{j \neq i} q_{ij}(f(j) - f(i)).$$

It determines Q uniquely, hence also the distribution of the process.
The transition rates fulfill

$$\begin{aligned} P(X_t = j | X_0 = i) &= q_{ij}t + o(t), i \neq j \\ P(X_t = i) | X_0 = i) &= 1 - q_i t + o(t). \end{aligned}$$

2.2 Probabilistic representation

Starting from Q we can realize the process in the following way. Let

$$p_{ij} = \frac{q_{ij}}{q_i}, i \neq j, \ p_{ii} = 0.$$

This is the transition matrix of a discrete-time Markov chain, the so-called embedded chain. When the chain is in state i, it stays there for a random holding time, exponentially distributed with parameter q_i, and then jumps to j according to p_{ij}. An equivalent description is the following. Assume $\sup_i q_i \leq q < \infty$ and define

$$r_{ij} = \frac{q_{ij}}{q}, i \neq j, \ r_{ii} = 1 - \sum_{j \neq i} r_{ij}.$$

When the chain is in state i, it stays there for a random holding time, exponentially distributed with parameter q, and then makes a transition to j

according to r_{ij}, so in this second realization the process may stay put in i with probability r_{ii}. This representation immediately shows how such a process may be simulated by simulation of exponentially distributed random variables and discrete transitions from i to j. It also shows the different roles of q_i and $\frac{q_{ij}}{q_i}$, the first determining the rate at which transitions occur and the second the probabilities according to which the new state is selected.

2.3 Representation using a random time transformation

We now assume that S is a subset of $\mathbb{Z}^d$; setting transition rates outside of S equal to zero, S may be assumed to be equal to $\mathbb{Z}^d$. The basic building block is the Poisson process which remains in any state $i = 0, 1, 2, \ldots$ with exponential holding time with parameter 1 and then jumps to $i + 1$. A continuous-time Markov chain $(X_t)_{t\in[0,\infty)}$ fulfills the following integral equation with $\gamma_l(i) = q_{i,i+l}$

$$X_t = X_0 + \sum_l lY^l(\int_0^t \gamma_l(X_s)ds)$$

where the $(Y_t^l)_t$ are independent Poisson processes and we write Y_t^l as $Y^l(t)$; see [6], 6.4. Here the clock for the Poisson process Y^l runs with the random speed $\int_0^t \gamma_l(X_s)ds$. We shall see in the following how this representation immediately sheds light on the limiting behaviour for such processes.

3 Basic facts on the asymptotic behaviour

Let us assume that for $N = 1, 2, \ldots$ we have continuous-time Markov chains $(Z_t^N)_t$ with state space S^N and transition rates q_{ij}^N, depending on N. In typical examples N is the size of a population or of a network. For large N, the exact behaviour is no longer tractable so one has to rely on approximations. In mathematical terms, the limiting behaviour of suitably standardized versions $(X_t^N)_t$ of $(Z_t^N)_t$ as $N \to \infty$ has to be investigated. There are various mathematical methods to achieve this.

3.1 Convergence via infinitesimal operators

Suppose that $S^N \subseteq S$ for all N. The convergence of the process $(X_t^N)_t$ can be reduced to a study of the convergence of the infinitesimal operators A^N. This is a general fact and does not depend on the assumption of discrete

state spaces which is discussed here.
Assume that we find an operator A, defined on a suitable set $\mathcal{D}$ of bounded functions $f : S \to \mathbb{R}$, where Af again is a function $Af : S \to \mathbb{R}$, such that the following condition holds

$$\lim_{N\to\infty} \sup_{y\in S^N} \left|A^N f(y) - Af(y)\right| \to 0.$$

If A generates a well-behaved Markov process $(X_t)_t$, well-behaved here in the mathematical sense of a Feller process, then

$$(X_t^N)_t \text{ converges to } (X_t)_t.$$

In exact mathematical term, this convergence takes place in the sense of weak convergence in the function space $D_S[0,\infty)$, which includes the convergence of finite-dimensional distributions and e.g. also convergence of stationary distributions and distributions of first hitting times. We refer to [6], 1.6.1, 4.2.11.
In many examples, A will be a differential operator on a certain set $\mathcal{D}$ of twice differentiable functions and the limit process will be a diffusion.

3.2 Convergence via the representation 2.3

For a concise discussion we let N denote the size of a population and $S^N = \{0, 1, \ldots, N\}$. Z_t^N describes the random number of a certain species within the population. The standardized version is given by the proportion of this species

$$X_t^N = \frac{Z_t^N}{N}$$

taking values in $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$. Now assume that the transition rates for $(Z_t^N)_t$ fulfill

$$\gamma_l^N(i) = q_{i,i+l}^N = N\beta_l(\frac{i}{N})$$

for some bounded function β on $[0,1]$, which may be generalized to $\gamma_l^N(i) = q_{i,i+l}^N = N(\beta_l(\frac{i}{N}) + O(\frac{1}{N}))$. This implies that the mean number of transitions in one unit of time is of the order N, equal in order to the size of the population. So we expect a law of large numbers to hold for X_t^N, hence a non-random limiting process $(X_t)_t$, and similarly a central limit theorem. This may be seen with the following arguments, and we refer to [6], 11.2 for a rigorous and complete discussion.

3.2.1 A law of large numbers

Let $\tilde{Y}_t = Y_t - t$ be a centered Poisson process. The law of large numbers for this process states that for all t

$$\sup_{s \leq t} \left| \frac{1}{N} \tilde{Y}_{Ns} \right| \to 0 \text{ almost surely for } N \to \infty.$$

Using the representation from 2.3 for $(Z_t^N)_t$ we have, with $f(x) = \sum_l l\beta_l(x)$,

$$X_t^N = \frac{1}{N} Z_t^N = X_o^N + \sum_l l \frac{1}{N} \tilde{Y}^l \left(N \int_0^t \beta_l(X_s^N) ds \right) + \int_0^t f(X_s^N) ds.$$

So our random clock has a speed of order N. Assuming $X_0^N \to x_0$ and Lipschitz continuity of f, the law of large numbers for the centered Poisson process implies for all t

$$\sup_{s \leq t} \left| X_s^N - X(s) \right| \to 0 \text{ almost surely for } N \to \infty.$$

where $X(t)$ is the non-random solution of the differential equation

$$\frac{d}{dt} X(t) = f(X(t)), X(0) = x_0.$$

3.2.2 A central limit theorem

The basic central limit theorem states that, for a sum of i.i.d. random variables X_i with finite mean μ and finite positive variance, the normalized sum $\frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - \mu)$ converges to a normal distribution. Writing

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N (X_i - \mu) = \sqrt{N} \left(\frac{1}{N} \sum_{i=1}^N X_i - \mu \right)$$

we see that the term $\frac{1}{N} \sum_{i=1}^N X_i - \mu$, which tends to zero according to the law of large numbers, has to be enlarged by a factor $\sqrt{N}$ to obtain a non-degenerate asymptotic normal distribution. This is a wide-spread phenomenon, so in our setting it is readily conjectured that the stochastic process $\left(\sqrt{N}(X_t^N - X(t)) \right)_t$ converges to a non-degenerate process which is Gaussian. To see this we use the central limit theorem for the centered Poisson process. This states that for the process $(W_t^N)_t$ with $W_t^N = \frac{1}{\sqrt{N}}(Y_{Nt} - Nt)$

$$(W_t^N)_t \text{ converges to a standard Wiener process } (W_t)_t.$$

Set

$$\begin{aligned} V_t^N &= \sqrt{N}(X_t^N - x(t)) = V_0^N + \sum_l lW^{l,N}(\int_0^t \beta_l(X_s^N)ds) + \int_0^t \sqrt{N}(f(X_s^N - f(X(s)))ds \\ &= V_0^N + \sum_l lW^{l,N}(\int_0^t \beta_l(X_s^N)ds) + \int_0^t \sqrt{N}\left(f(X(s) + \frac{1}{\sqrt{N}}V_s^N) - f(X(s))\right)ds. \end{aligned}$$

Letting N tend to infinity the limiting equation looks formally, with $V_0^N \to v_0$,

$$V_t = v_0 + U_t + \int_0^t f'(X(s))V_s ds$$

where $U_t = \sum_l lW^l(\int_0^t \beta_l(X(s))ds)$ is a Gaussian process. It can be proven rigorously that in fact

$$(\sqrt{N}(X_t^N - X(t)))_t \text{ converges to a process } (V_t)_t$$

with $(V_t)_t$ a solution of the limiting equation. Since $(U_t)_t$ is a Gaussian process, $(V_t)_t$ also is a Gaussian process. Gaussian processes are uniquely determined by their mean and covariance function. From the limiting equation one can obtain the following expressions:
Let $g(t,s)$ be a solution of

$$\frac{d}{dt}g(t,s) = f'(X(t))g(t,s), \ \ g(s,s) = 1.$$

Then the mean of V_t is given by $g(t,0)v_0$ and the covariance is given by

$$Cov(V_t, V_r) = \int_0^{\min\{t,r\}} g(t,s)g(r,s)\sum_l l^2\beta_l(X(s))ds.$$

So we obtain that the process

$$(X_t^N)_t \text{ is approximated by } X(t) + \frac{1}{\sqrt{N}}V_t,$$

hence

$$(Z_t^N)_t \text{ is approximated by } (NX(t) + \sqrt{N}V_t)_t.$$

The distribution of Z_t^N is approximated by a normal distribution with mean $Nx_0 + \sqrt{N}v_0$ and variance $NVar(V_t)$.

4 Agent-based models and their asymptotic behaviour

In agent-based models we have a population of N agents interacting on a financial market. These agents are of two different types, e.g. optimists and pessimists. The number of agents of one type, e.g. optimists, is described by a continuous-time Markov chain $(Z_t^N)_t$ with state space $\{0, 1, \ldots, N\}$. Agents may switch from one type to the other, and it is the usual assumption that $(Z_t^N)_t$ follows a birth and death process where

$$q_{i,i+l} = 0 \text{ for } |l| > 1.$$

A birth here means the conversion of a pessimist to an optimists, a death the opposite conversion. The birth rate, and death rate respectively, are

$$\lambda_i = q_{i,i+1} \text{ for } i = 0, 1, \ldots, N-1,\ \ \mu_i = q_{i,i-1} \text{ for } i = 1, \ldots, N,$$

with $\lambda_N = \mu_0 = 0$. For a birth and death process model the probability of multiple switches in time interval of length t tends to zero as t tends to zero. There are two main models which have been proposed in the literature, let us call them model 1 and model 2. We refer to [4] for a thorough discussion. Model 1 looks at the birth and death rates

$$\lambda_i = (N-i)(a + b\frac{i}{N}),\ \ \mu_i = i(a + b\frac{N-i}{N}),$$

Model 2 looks at the birth-and-death rates

$$\lambda_i = (N-i)(a + bi),\ \ \mu_i = i(a + b(N-i))$$

where a describes the overall tendency to switch, b the tendency due to group pressure.

4.1 Analysis of model 1

Referring to the discussion in Section 2, we slightly generalize the model to $\lambda_i = (N-i)(a_1 + b_1\frac{i}{N}),\ \mu_i = i(a_2 + b_2\frac{N-i}{N})$ and have with $y = \frac{i}{N}$

$$\begin{aligned} q_i &= \lambda_1 + \mu_i = N\left[(1-y)(a_1 + b_1 y) + y(a_2 + b_2(1-y))\right] \\ p_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i} = \frac{1}{1 + [(1-y)(a_1 + b_1 y)]^{-1} y(a_2 + b_2(1-y))}, \\ p_{i,i-1} &= 1 - p_{i,i+1}. \end{aligned}$$

So we see that the expected number of switches, which may be loosely interpreted as being proportional to the expectet number of encounters of agents in one unit of time is of the order N. The probabilities $p_{i,i+1}$ and $p_{i,i-1}$ which describe into which direction switching occurs, only depend on the fraction of agents of one type in the population. The limiting behaviour of the normalized process $X_t^N = \frac{Z_t^N}{N}$ follows readily from 3.2. Using the notation from this section we have

$$\gamma_l^N(i) = q_{i,i+l} = N\beta_l(\frac{i}{N}),\ l = -1, 1,$$

with $\beta_1(x) = (1-x)(a_1+b_1x)$, $\beta_{-1}(x))x(a_2+b_2(1-x))$ defined for $x \in [0,1]$. From this it is clear that 3.2 applies. We have

$$f(x) = \beta_1(x) - \beta_{-1}(x) = (1-x)(a_1+b_1x) - x(a_2+b_2(1-x)).$$

The limiting process is the non-random solution of the differential equation, which is an inhomogeneous Bernoulli-differential equation,

$$\frac{d}{dt}X(T) = f(X(T)),\ X(0) = x_0.$$

As we have seen in 3.2, the standardized differences $\sqrt{N}(X_t^N - X(t))$ tend to a Gaussian process V_t. Assume $V_0 = 0$, the mean value function is zero. We have $f'(x) = b_1 - b_2 - a_1 - a_1 + 2(b_2 - b_1)x$, and the covariance function may be computed, at least numerically, from the linear differential equation

$$\frac{d}{dt}g(t,s) = f'(X(t))g(t,s),\ g(s,s) = 1,$$

and the resulting expression in 3.2. Let us now make the following standard assumption $b_1 = b_2 = b$. Then

$$f(x) = (a_1+a_2)\left(\frac{a_1}{a_1+a_2} - x\right),$$

and it follows

$$X(t) = \frac{a_1}{a_1+a_2} - \left(x_0 - \frac{a_1}{a_1+a_2}\right)e^{-(a_1+a_2)t}.$$

This shows that $X(t)$ converges to $\frac{a_1}{a_1+a_2}$ as $t \to \infty$.
We have $f'(x) = -(a_1+a_2)$ and a solution of the above differential equation is given by

$$g(t,s) = e^{-(a_1+a_2)(t-s)}.$$

It follows

$$\begin{aligned}
& Cov(V_t, V_r) \\
= & \int_0^{\min\{t,r\}} e^{-(a_1+a_2)(t+r-2s)}((2b-a_1+a_2)s-2bs^2+a_1)ds \\
= & \frac{e^{-(a_1+a_2)(t+r)}}{2(a_1+a_2)} \int_0^{\min\{t,r\}\cdot 2(a_1+a_2)} e^{-s}(a_1+\frac{2b-a_1+a_2}{2(a_1+a_2)}s-\frac{2b}{4(a_1+a_2)^2}s^2)ds \\
= & \frac{e^{-(a_1+a_2)(t+r)}}{2(a_1+a_2)} \Bigg[a_1+\frac{2b-a_1+a_2}{2(a_1+a_2)}+2\frac{2b}{4(a_1+a_2)^2} \\
& -e^{-\min\{t,r\}\cdot 2(a_1+a_2)}\Bigg(a_1+\frac{2b-a_1+a_2}{2(a_1+a_2)}(1+\min\{t,r\}\cdot 2(a_1+a_2)) \\
& +\frac{2b}{4(a_1+a_2)^2}(2+2(\min\{t,r\}\cdot 2(a_1+a_2))+(\min\{t,r\}\cdot 2(a_1+a_2))^2)\Bigg)\Bigg]
\end{aligned}$$

4.2 Analysis of model 2

In model 2 we have, with slight generalization and $y=\frac{i}{N}$

$$\begin{aligned}
\lambda_i &= (N-i)(a_1+b_1 i) = N^2(1-y)\left(\frac{a_1}{N}+b_1 y\right), \\
\mu_i &= i(a_2+b_2(N-i)) = N^2 y\left(\frac{a_2}{N}+b_2(1-y)\right)
\end{aligned}$$

with

$$\begin{aligned}
q_i &= \lambda_i+\mu_i = N^2\left[(1-y)\left(\frac{a_1}{N}+b_1 y\right)+y\left(\frac{a_2}{N}+b_2(1-y)\right)\right] \\
p_{i,i+1} &= \frac{1}{1+\left[(1-y)\left(\frac{a_1}{N}+b_1 y\right)\right]^{-1}\frac{i}{N}\left(\frac{a_2}{N}+b_2(1-y)\right)}.
\end{aligned}$$

There are two changes with regard to model 1. The expected number of switches in one time period has increased to the order of N^2, and the overall tendency of switching has decreased to the order of $\frac{1}{N}$. This, of course, is a dramatic chance, as e.g. for $N=100$, the number of switches in one unit of time is no longer of the order 100 but of the order 10.000. Due to the first change it is clear that a law of large number as stated in 4.1 for model 1 can no longer hold. To analyse the asymptotic behaviour of the system we use the method of infinitesimal operators as described in 3.1. Let $X_t^N=\frac{Z_t^N}{N}$ as before and denote the infinitesimal operator of $(X_t^N)_t$ by A^N. It follows for $y=\frac{i}{N}, 0<\frac{i}{N}<1$

$$A^N f(y) = \lambda_i(f(y+\frac{1}{N})-f(y))+\mu_i(f(y-\frac{1}{N})-f(y)).$$

Let $\mathcal{D}$ denote the set of continuous mappings $f : [0,1] \to \mathbb{R}$ which are twice-continuous differentiable in the interior such that the derivatives have a continuous extension to 0 and 1. A Taylor expansion for $0 < y < 1$ shows

$$A^N f(y) = \frac{1}{N}(\lambda_i - \mu_i)f'(y) + \frac{1}{2N^2}(\lambda_i + \mu_i)f''(y) + (\lambda_1 + \mu_i)o(\frac{1}{N^2})$$

with error term $o(\frac{1}{N^2})$ uniform in y. Inserting the birth-and-death rates of model 2 we find

$$\begin{aligned}
A^N f(y) &= \frac{1}{N}N^2\left((1-y)(\frac{a_1}{N} + b_1 y) - y(\frac{a_2}{N} + b_2(1-y))\right)f'(y) \\
&+ \frac{1}{N^2}\frac{N^2}{2}\left((1-y)(\frac{a_1}{N} + b_1 y) + y(\frac{a_2}{N} + b_2(1-y))\right)f''(y) + o(1) \\
&= N\left((1-y)\frac{a_1}{N} - y\frac{a_2}{N} + b_1 y(1-y) - b_2 y(1-y)\right)f'(y) \\
&+ \frac{1}{2}\left((1-y)(\frac{a_1}{N} + b_1 y) + y(\frac{a_2}{N} + b_2(1-y))\right)f''(y) + o(1).
\end{aligned}$$

From this it is clear that $A^N f$ cannot converge for $b_1 \neq b_2$ and also not for some fixed overall switching tendencies $\tilde{a}_1, \tilde{a}_2$ which are not of order $O(\frac{1}{N})$. So in the case of the standard assumption $b_1 = b_2 = b$ we have

$$\begin{aligned}
A^N f(y) &= ((1-y)a_1 - ya_2)f'(y) + \frac{1}{2}\left(2b(1-y)y + \frac{a_1}{N}(1-y) + \frac{a_2}{N}y\right)f''(y) + o(1) \\
&\to ((1-y)a_1 - ya_2)f'(y) + \frac{1}{2}2b(1-y)yf''(y) \text{ as } n \to \infty.
\end{aligned}$$

If A is the differential operator on $\mathcal{D}$ defined by the last line, then we see that

$$\sup_{y \in S^N, y \neq 0,1} |A^N f(y) - Af(y)| \to 0 \text{ as } N \to \infty.$$

The convergence of $A^N f(0) \to Af(0)$, $A^N f(1) \to Af(1)$, where Af is continuously extended to $0, 1$, requires in addition $f'(0) = f'(1) = 0$. But the set of all functions in $\mathcal{D}$ with this additional property is rich enough to yield convergence of the process according to 3.1, see [8]. So $(X_t^N)_t$ converges to a diffusion on $[0,1]$ with drift term $\mu(y) = (1-y)a_1 - ya_2$ and diffusion term $\sigma^2(y) = 2by(1-y)$.

5 Mean field analysis of agent networks

If we think of agents communicating in a network we can model transition rates through a term which represents a general switching tendency and a term which depends on the neighbours with respect to the particular network structure. So for any agent α, the transition rate to switch from a state k to a state l might be modelled as a function

$$t_\alpha(k, l; N_\alpha(k), N_\alpha(l), N)$$

where N is the total number of agents in the network, $N_\alpha(k)$ is the number of neighbours of agent α in state k, $N_\alpha(l)$ is the number of neighbours in state l. Of course, more complicated models are possible involving neighbouring agents in different states, but already a model of the above type is analytically intractable on the individual agent level. So one may resort to an analysis of the conglomerated number of agents in a particular state k with transition rates depending on the average number of agents that any agent is linked to and their states; see [5] for a derivation within an economic context. Going back to two states, e.g. optimistic and pessimistic agents in the network, the number Z_t^N of optimistic agents may be viewed as a birth and death process. We can model the birth and death rates, with $y = \frac{i}{N}$, as

$$\begin{aligned}
\lambda_i &= h(N)(1-y)(a_1(N) + b_1(N)y), \\
\mu_i &= h(N)y(a_2(N) + b_2(N)(1-y)).
\end{aligned}$$

Here $a_i(N)$ is the overall switching tendency in the network and $b_i(N)$ describes the group pressure depending on the mean number of neighbours in the network. $h(N)$ gives the order of the mean number of switches within one unit of time, loosely interpreted as being of the order of the mean number of encounters in the network. All the parameters $h(N), a_i(N), b_i(N)$ can depend on the network topology.

Suppose that any agent is linked to a fixed proportion γ of the other agents in the network and the mean number of encounters in one unit of time is of the order δN. Then, with $a_i(N) = a_i$ and $b_i(N) = \gamma b_i$ we arrive at model 1 of Section 4

$$\begin{aligned}
\lambda_i &= \delta N(1-y)(a_1 + \gamma b_1 y), \\
\mu_i &= \delta N y(a_2 + \gamma b_2(1-y)).
\end{aligned}$$

Assume that the mean number of encounters in unit time increases to δN^2. Then with $a_1(N) = \frac{a_1}{N}$ and $a_2(N) = \frac{a_2}{N}$ we arrive at

$$\begin{aligned}
\lambda_i &= \delta N^2(1-y)(\frac{a_1}{N} + \gamma b_1 y), \\
\mu_i &= \delta N^2 y(\frac{a_2}{N} + \gamma b_2(1-y)),
\end{aligned}$$

which is model 2 of Section 4.
Now assume that the number of agents to which any agent is linked grows slower than the total agent number N. Then we may model the proportion of linked agents as $\gamma f(N)$ where $f(N)$ tends to 0 as N tends to infinity. So we arrive at

$$\begin{aligned}\lambda_i &= h(N)(1-y)(a_1(N)+\gamma f(N)b_1 y),\\ \mu_i &= h(N)y(a_2(N)+\gamma f(N)b_2(1-y)).\end{aligned}$$

By a suitable choice of $a_i(N)$ and $h(N)$ we can again arrive at model 1 or model 2. Speeding e.g. up the agent encounters to $h(N)=\frac{N^2}{f(N)}$ with overall switching tendency $a_i(N)=\frac{a_i f(N)}{N}$ we arrive at model 2

$$\begin{aligned}\lambda_i &= N^2(1-y)(\frac{a_1}{N}+\gamma b_1 y),\\ \mu_i &= N^2 y(\frac{a_2}{N}+\gamma b_2(1-y)).\end{aligned}$$

The conclusion is that network topology and speed of communication within the network work together to obtain a limiting behaviour which mimics the behaviour of financial markets.

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