Integrating Real Sector Growth and Inflation Into An Agent-Based Stock Market Dynamics

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May 2014

Abstract

Concentrating on speculative flow rather than stock demand, the paper puts forward a deterministic continuous-time model of the equity market that is compatible with a growing and inflationary economy. Instead of the systematically rising equity price, the central state variable is now Tobin's q, which makes it necessary to consider explicitly the financing of fixed investment in the real sector. Integrating a number of suitable re-specifications and fixing the variables in the real sector, the model succeeds in re-establishing (almost) the same mathematical structure as the elegant two-dimensional Lux (1995) model, which implicitly was set up in the usual stationary and non-inflationary environment. Thus a speculative dynamics is obtained that can generate persistent oscillations as well as bubble equilibria and a rich sequence of local and global bifurcations. The model is ready to be combined with the growth cycles in a real sector, where the short-term fluctuations of Tobin's q may then also affect aggregate demand.

JEL classicification: D 21, D 84, G 12.

Keywords: Speculative dynamics, stock price inflation, herding, logit dynamics, bifurcations.

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1 Introduction

In a long-term perspective, this paper seeks to contribute to an integration of a speculative stock market dynamics into models of the real sector that exhibit growth cycle behaviour and positive inflation. The problem that we perceive is with the modelling of the stock market. While in this respect three main approaches may be distinguished in the existing literature, we know of no fully satisfactory example that could be directly adopted for such an integration.

A first approach can be viewed as an extension of the old textbook LM framework to more financial assets than just money and bonds, two seminal articles being Brainard and Tobin (1968) and Tobin (1982).[1] Subsequent work has shown its elegance and that growth and inflation pose no particular problems; see, e.g., Taylor and O'Connell (1985) or Asada et al. (2010). On the other hand, the specification of demand on the financial markets as the agents' desired holdings—as opposed to the desired *changes* in their positions—is not a convincing foundation for a theory of speculation.[2] In addition, the typical architecture of these models sets up links between the real and financial sector which are so tight that all variables would oscillate with the same period (clearly demonstrated in Franke and Semmler, 1999).

Originating with Blanchard (1981), a second line of research puts more emphasis on the expectations about future capital gains and so, dependent on the specific assumptions and, in particular, their nonlinearities, could give rise to some volatility (an introduction into this kind of modelling is Chiarella et al., 2009, Chapters 2, 6, 7). An explicit representation of demand and supply of the financial assets is here, however, absent; by way of arbitrage arguments, share prices are assumed to react instantaneously or with a delay to the differential between expected equity returns and the rate of interest. Heterogeneity of agents is possible in this setting, but only with respect to the capital gains expectations (Chiarella et al., 2009, p. 221), not with respect to different principles of the formulation of demand.

With the archetypes of fundamentalist and chartist traders, the latter is indeed at the heart of the many agent-based asset pricing models that have been developed since Beja and Goldman (1980).[3] There are also a few recent papers that combine such a model with a real sector (Lengnick and Wohltmann, 2011, and Westerhoff, 2012).[4] Unfortunately, despite its rich-

[1] Although LM may also refer to equities, this has no further consequences since they and bonds are assumed to be perfect substitutes.

[2] Taylor and O'Connell (1985, p. 875) explicitly state that their "wealthholders try to look through Wall Street".

[3] For surveys of this burgeoning field of research, see Hommes (2006), Lux (2009) and Westerhoff (2009), to name a few.

[4] Proaño (2011) has set up a model with interactions between a real sector and an

ness, all of this literature explicitly or implicitly refers to a stationary real sector without inflation. For example, the fundamental value is (occasionally) an exogenous random walk or (most often) a constant derived from a constant stream of dividends.[5]

Given the limitations of the first two approaches, the stationarity assumption in the otherwise very fruitful agent-based modelling presents a challenge that it is high time to accept. We therefore take a well-established agent-based model from the shelf, for which we choose the two-dimensional speculative dynamics put forward by Lux (1995) because of its elegance and its potential for persistent cycles and locally stable bubble equilibria. It is our aim to respecify the model such that it can account for growth and inflation in the real sector, and such that the original structure is essentially preserved.

In this new context, the stock prices and their fundamental value have to be converted into a stationary variable, which means they have to be scaled by goods prices and a measure of economic activity. In this way Tobin's q as the ratio of stock market valuation and the replacement value of the capital stock will be one of our key variables. For its theoretical underpinning we propose a structural framework with corporate firms in its centre who may issue equities to finance part of their fixed investment, or who may systematically buy them back from the market. As it will turn out, a classical value $q^\star = 1$ could again make perfect economic sense as an objective fundamental value, although due to heterogeneous perceptions of the traders it need not necessarily constitute a long-run equilibrium. In any way, the determination of Tobin's q on the stock market is influenced by some variables from the real sector, while regarding our perspective mentioned at the beginning, it could also be easily employed as a variable affecting aggregate demand on the goods market, which would give us a feedback channel in the opposite direction.

The remainder of the paper is organized as follows. Section 2 describes the single components of the stock market model. Treating the real sector as exogenously given, Section 3 puts them together and thus arrives at a two-dimensional differential equations system that shows a close resemblance to the Lux model. Three propositions summarize what can be mathematically inferred about its dynamic properties. Section 4 is concerned with a numerical study. It puts up a benchmark scenario for persistent oscillations, studies the impact of a *ceteris paribus* variation of a herding parameter on

agent-based exchange rate market.

[5]A certain exception is Chiarella et al. (2013), who principally allow the dividends to grow over time. However, their price equation (which is only sloppily explained) takes no account of possible changes in the number of shares, and persistent stock price inflation would imply persistent excess demand, so that the so-called market maker would run out of his inventory.

the dynamics, which gives rise to a rich sequence of local and global bifurcations (richer than in the original Lux model), and finally takes a quick look at the motions induced by some regular exogenous oscillations in the real sector. Section 5 concludes. A number of finer details are relegated to five appendices.

2 The stock market model

2.1 The financing of fixed investment and Tobin's q

Since we do not want to limit ourselves to a constant stock of outstanding shares, we need to consider the issuance policy of the firms in the real sector and its connection to the dividends payed out to the shareholders. These components are linked through the finance equation according to which fixed investment is financed by a combination of internal and external sources. Regarding the former, a fraction σ_f of the firms' profits are retained, while the issuing of new shares is supposed to be the only external source. Assuming a one-good economy, profits are given by $r\,pK$, where K denotes the capital stock in place, p the general price level, and r the rate of profit. The number of shares is designated E, p_e is their current price, and $\dot{E}$ their instantaneous rate of change.[6] With I the level of net investment, the finance identity in continuous time reads,

$$pI \;=\; \sigma_f\, r\, pK \;+\; p_e \dot{E} \tag{1}$$

Firms need not permanently issue new shares; for a certain period of time they may also buy them back from the market, so that both $\dot{E} \geq 0$ and $\dot{E} < 0$ are admissible. Likewise, the retention rate σ_f need not necessarily be nonnegative; it might fall below zero if $\dot{E}$ is high enough. In any case, however, if besides investment and the profit rate, it is $\dot{E}$ that is predetermined by some rule, σ_f is residually determined and *vice versa*. Following much of the macroeconomic literature, we assume a hierarchy where external finance takes a higher level than internal finance. Correspondingly, the rate of growth g_e of equities ($g_e = \dot{E}/E$) is treated as a predetermined variable.[7] Introducing the capital growth rate $g = I/K$ and (average) Tobin's q,

$$q \;=\; \frac{p_e E}{pK} \tag{2}$$

and noting that $p_e\dot{E}/pK = (p_e E/pK)\,(\dot{E}/E) = q\,g_e$, it is easily seen from solving (1) for σ_f that the retention rate varies with Tobin's q,

$$\sigma_f \;=\; \sigma_f(q) \;=\; (g - q\,g_e)\,/\,r \tag{3}$$

[6] For a dynamic variable x we write $\dot{x} = dx/dt$ for its derivative with respect to time and, further below, $\hat{x} = \dot{x}/x$ for its instantaneous growth rate.

[7] Appendix A proposes a straightforward rule for choosing g_e.

Whether the relationship is positive or negative depends on the sign of g_e. Incidentally, eq. (3) can be interpreted as a variant of the Cambridge equation, which is here adjusted to a framework where part of investment is financed by issuing equities.

The changes of Tobin's q are readily obtained from logarithmic differentiation, $\hat{q} = \hat{p}_e + \hat{E} - \hat{p} - \hat{K}$. With the definition of the growth rates g_e and g and writing π for the rate of inflation in the real sector, they read,

$$\hat{q} \;=\; \hat{p}_e + g_e - \pi - g \tag{4}$$

A state of the stock market where share prices change such that Tobin's q remains constant will be of particular importance. This special value of $\hat{p}_e$ (when g, g_e and π are predetermined) provides an objective benchmark for stock price inflation, which may be designated $\pi_e^\star$,

$$\pi_e^\star := \hat{p}_e|_{\hat{q}=0} \;=\; \pi + g - g_e \tag{5}$$

The equation tells us that in the presence of $g_e < g$, stock prices tend to rise faster than goods prices. with positive inflation in the real sector, $\pi > 0$, $\pi_e^\star$ will be strictly positive, and $\pi_e^\star = \pi$ prevails if and only if the number of shares increases at the same rate as the real capital stock.

Equities themselves are held because of the dividend payments $(1-\sigma_f)\,r\,pK$ and the capital gains $\hat{p}_e$. Thus, taking account of (3), the instantaneous (or 'daily') rate of return on equities, r_e, is given by

$$r_e \;=\; \frac{(1-\sigma_f)\,r\,pK}{p_e E} + \hat{p}_e \;=\; \frac{r-g}{q} + g_e + \hat{p}_e \tag{6}$$

2.2 The price adjustment equation

Regarding the principle governing the evolution of stock prices, let us start from the basic idea adopted in so many asset pricing models that there is a so-called market maker who raises (lowers) the price if demand exceeds (falls short of) supply. The present context, however, requires a modification of the standard specification when inflation in the real sector is neglected. This is most directly seen when considering the notion of an equilibrium position, where Tobin's q should be constant.

From eq. (5) we know that stock price inflation will then be typically positive. The standard approach would therefore imply a permanent excess demand on the market. According to the usual story, the excess of demand over supply is served by the market maker, so that sooner or later he would run out of his inventory and this mechanism breaks down (presupposing that he would not be willing to replenish the inventory at his own cost). Hence, in order to guard against this tendency, the market maker would have to increase the market price even when demand and supply balance. To this end, we assume that he continues to adjust prices in the direction of excess demand but, in addition, employs a benchmark rate $\tilde{\pi}_e^m$ of stock price inflation. Furthermore, a possible longer growth or decline of equities is taken into account by normalizing the volume of excess demand by the number of outstanding shares. In this way, the price equation becomes

$$\hat{p}_e \;=\; \tilde{\pi}_e^m \;+\; \tilde{\beta}_e \times \text{total excess demand} \,/\, E \tag{7}$$

where $\tilde{\beta}_e$ is a positive price impact factor (the two tildes will be removed shortly below, after a convenient transformation). The market maker's benchmark inflation $\tilde{\pi}_e^m$ may or may not be equal to the objective benchmark $\pi_e^\star$ from eq. (5).

There is a great variety of agent-based models working with just two groups of speculative traders (though most of them are formulated in discrete time). One of them are the fundamentalists, who bet on an eventual return of the market price to some fundamental value p_e^f and thus buy (sell) if p_e is below (above) p_e^f. Now neglect the other traders for a moment and suppose a positive growth rate of equities. In an equilibrium the fundamentalists would consequently have to have a positive excess demand in order to absorb them, which in turn means market prices prevail that are persistently below their fundamental values (both of which may increase over time). This type of behaviour which supports an equilibrium with persistent misalignment may not appear fully consistent and we would have to reveal a second motive to make sense out of it. An analogous reasoning would apply to the second group of traders, depending on their specific strategies.

We prefer to make such an additional motive explicit and attribute it to an extra group of traders.[8] Their actions are related to absorbing the shares newly issued if $g_e > 0$, and to provide shares if firms are buying them back, $g_e < 0$. We may call these agents short-term fundamentalists, in contrast to the 'normal' fundamentalists with their medium-term trading horizon. The basic idea is as follows. If stock prices currently change at a rate such that the resulting equity rate of return r_e is equal to the (risk-adjusted) rate of interest, then the short-term fundamentalists are indifferent between buying or selling shares. More specifically, under these circumstances their demand is just equal to the change in equities, $\dot{E}$. If the returns r_e are higher (lower) than the bond rate, their demand is proportionately higher (lower) than in the situation just described.

The idea is formulated in detail in Appendix B. It is an advantage of our continuous-time framework that the original price equation (7) can then be simplified such that the short-term fundamentalists remain in the back-

[8] Of course, a single trader may split up his capital (and his personality, so to speak) and simultaneously follow different rules.

ground. In the total excess demand on the market, the demand for and supply of new equities cancels and what is left is just the demand of the truly speculative traders. Taking up the types of traders considered by Lux (1995) and adjusting them somewhat to the present setting, these are the fundamentalists with (net) demand d^f and the so-called sentiment traders with (net) demand d^s, where d^f and d^s may be positive as well as negative. With a new benchmark rate of stock price inflation π_e^m, which results from a suitable transformation of $\tilde{\pi}_e^m$, and a new positive price impact factor β_e, which results from a suitable transformation of the original $\tilde{\beta}_e$ in (7), we can then work with the following, more convenient, price adjustment equation,

$$\hat{p}_e \quad = \quad \pi_e^m \; + \; \beta_e \, (d^f + d^s) \, / \, E \tag{8}$$

(compare this equation with eq. (A1) in Appendix B.) Under a certain and not too implausible condition, π_e^m in this reduced form is given by the market maker's benchmark $\tilde{\pi}_e^m$ in (7). We will nonetheless be indifferent in this regard and, for simplicity, refer directly to the present π_e^m as the market maker's benchmark for stock price inflation.

2.3 Fundamental value and fundamentalists

The fundamentalists in the usual sense, identified by a superscript 'f', have longer time horizons and base their demand on the differences between the current price and what they perceive as the fundamental value. Even though they might expect the gap between the two prices to widen in the immediate future, they do not trade on these short-run expectations. Instead they choose to place their bets on an eventual rapprochement.

The fundamental value adopted in the formulation of their demand derives from the equity rate of return r_e^f that is relevant for them. The latter means that they use a smoother measure of the capital gains in its specification than the rather volatile values of $\hat{p}_e$. In fact, they replace the instantaneous capital gains $\hat{p}_e$ in (6) with a constant benchmark rate π_e^f, that is,

$$r_e^f \;=\; \frac{(1-\sigma_f)\, r\, pK}{p_e E} \;+\; \pi_e^f \;=\; \frac{r-g}{q} \;+\; g_e \;+\; \pi_e^f \tag{9}$$

The fundamentalist traders are indifferent between holding equities or government bonds if this rate of return is equal to the interest rate i plus a risk premium ξ. Equality is brought about by a suitable value of Tobin's q. Solving the equation $r_e^f = r_e^f(q) = i + \xi$ for q, this fundamental q^f is derived from π_e^f as

$$q^f \;=\; \frac{r-g}{i+\xi-g_e-\pi_e^f} \tag{10}$$

and the corresponding fundamental price is

$$p_e^f \;=\; q^f\, pK/E \tag{11}$$

Fundamentalists view equities as undervalued and therefore have a positive demand for them if the market price p_e is presently below this fundamental value p_e^f, and a negative demand if presently p_e exceeds p_e^f.[9] Letting $\tilde{\beta}_f$ be a measure of their general aggressiveness and/or their capital in the market, we posit the simple rule $d^f/E = \tilde{\beta}_f\,(p_e^f - p_e)/p_e^f$. This expression can be rewritten as $\tilde{\beta}_f\,(p_e^f E/pK - p_e E/pK)/(p_e^f E/pK) = \tilde{\beta}_f\,(q^f - q)/q^f$. Since in the present paper the variables determining q^f are treated as fixed data, we may put $\beta_f = \tilde{\beta}_f/q^f$. The demand of the fundamentalists is thus given by

$$d^f/E \;=\; \beta_f\,(q^f - q) \tag{12}$$

If in future work our stock market is combined with the business cycles in the real sectors so that q^f in (10) is varying over time, it might be convenient to postulate (12) directly.

It should be emphasized that the concept of r_e^f, q^f and p_e^f rests on a subjective benchmark π_e^f of stock price inflation. In particular, apart from possible misperceptions, the specific value of p_e^f may depend on the trading horizon of the fundamentalists. On the other hand, from a long-term perspective (which might be longer than the perspective of the fundamentalists) an appropriate benchmark would be the rate $\hat{p}_e = \pi_e^\star$ from (5) at which Tobin's q does not change. Substituting it for π_e^f in (10) gives us a value $q^\star$ that might seem a natural candidate for a long-run equilibrium value of Tobin's q,

$$q^\star \;=\; \frac{r - g}{i - \pi + \xi - g} \tag{13}$$

It is also the most natural value if it is taken into account that consistency of a steady state position in the real sector requires the profit rate to be equal to the real interest rate plus the risk premium, $r = i - \pi + \xi$. On this basis, the familiar value $q^\star = 1$ is obtained. Nevertheless, it will be seen that $q^\star$ may not necessarily constitute a point of rest for the stock market. Even the equality of $q^\star$ and the subjective benchmark q^f of the fundamentalists will not be sufficient for this.

2.4 The sentiment traders

The second group of the (truly) speculative agents are the sentiment traders. These are agents who are either optimistic or pessimistic, and who can

[9] They do not expect p_e to return to the current value of p_e^f, but that the gap between the two will narrow; for example, if p_e^f as well as p_e tend to rise in the near future.

probabilistically switch between the two attitudes. If a^+ denotes the share of optimists and a^- the share of pessimists (i.e., $a^+ + a^- = 1$), the difference a between the two may define the so-called sentiment index, $a = a^+ - a^-$. By construction, $-1 \le a \le 1$, and there are as many optimists as pessimists if the index is balanced, $a = 0$.

It is assumed that an average sentiment trader buys, if he is optimistic, a fixed (tiny) fraction $\tilde{\beta}_s$ of the shares currently outstanding, and he sells it if he is pessimistic. N_s being the size of the group, $a\, N_s\, \tilde{\beta}_s\, E$ represents their total demand d^s on the market. Putting $\beta_s = N_s\, \tilde{\beta}_s$,

$$d^s/E \;=\; \beta_s\, a \tag{14}$$

The law of motion for the sentiment index a is basically dependent on a so-called indicator of optimism z_a, higher values of which will be conducive to optimism. Specifically, it incorporates two effects. The first is a herding effect, meaning that a higher share of optimists make it more likely for an agent to remain, or become, an optimist. Let the strength of this effect be measured by a coefficient ϕ_a. The second effect is a trend effect, where in the same sense optimism is fostered by excess capital gains. The latter are assessed by a group-specific benchmark rate π_e^s, and the strength of this mechanism is governed by a coefficient ϕ_p. In sum, the indicator of optimism is set up as

$$z_a \;=\; \phi_a\, a \;+\; \phi_p\, (\hat{p}_e - \pi_e^s) \tag{15}$$

and $-z_a$ can be regarded as an indicator of pessimism. Equation (15) is the direct counterpart of eq. (9) in Lux (1995, p. 887).[10]

It remains to specify how z_a acts on the sentiment index a. Here it is assumed that a sentiment trader makes a new decision about his attitude with a uniform and fixed probability ν per unit of time, and with probability (1–ν) he sticks to his current attitude. If active in this way, a trader, irrespective of his previous behaviour, chooses to be an optimist with probability $P^+(z_a) := \exp(\gamma z_a)/[\,\exp(\gamma z_a) + \exp(-\gamma z_a)\,]$, and to be a pessimist with the complementary probability $P^-(z_a) := \exp(-\gamma z_a)/[\,\exp(\gamma z_a) + \exp(-\gamma z_a)\,]$. Interpreting the positive coefficient γ as the 'intensity of choice', these expressions are well-known from discrete choice theory. As explained in Appendix C, they lead to changes $\dot{a}^+ = \nu\,[P^+(z_a) - a^+]$ and $\dot{a}^- = \nu\,[P^-(z_a) - a^-]$, a so-called logit dynamics. Subtracting the two derivatives, we arrive at the following differential equation for the sentiment index,

$$\dot{a} \;=\; \nu\,[\,\tanh(\gamma z_a) - a\,] \tag{16}$$

The hyperbolic tangent in (16) is defined on the entire real line with tanh(0) = 0, it is everywhere increasing, skew-symmetric around zero, and bounded

[10]His discussion points out that the coefficient ϕ_p incorporates a time dimension.

between -1 and $+1$, where these values are asymptotically approached by $\tanh(x)$ as $x \to -1$ and $x \to +1$, respectively. A pleasant implication of this nonlinear shape is that if the indicator of optimism z_a remains bounded, the boundaries $a = \pm 1$ are repelling ($\tanh(\gamma z_a)$ eventually falls short of a and thus renders $\dot{a} < 0$ as $a \to +1$, and analogously for $a \to -1$). Hence there will always be a nondegenerate mix of optimists and pessimists.

It may be noted that eq. (10) in Lux (1995, p. 888), adjusted to the present context and notation, reads $\dot{a} = \nu\,[\tanh(\gamma z_a) - a]\,\cosh(\gamma z_a)$. Since the hyperbolic cosine is symmetric around zero and everywhere not less than one, the information content of the two adjustment equations is quite the same, although Lux derives his sentiment dynamics from a different specification based on endogenously varying transition probabilities.[11] We prefer to work with (16) since it need not 'bother' about the somewhat 'distorting' cosine term.

3 Analysis of the model

3.1 The functioning of the stock market dynamics

In order to study the speculative dynamics on the stock market in its pure form, we put the variables in the real sector to rest and treat r, i, π, ξ, g, g_e as fixed parameters. Before turning to a formal analysis, let us first consider the general functioning of the market. As a starting point, suppose Tobin's q is below the fundamentalist benchmark q^f and the mood of the sentiment traders is balanced, $a = 0$. Thus fundamentalists have a positive demand according to (12) and the sentiment traders are temporarily inactive according to (14). Stock prices consequently rise at a higher rate than the market maker's benchmark π_e^m; see eq. (8). Let us furthermore, for the sake of the argument, assume that π_e^m equals the objective benchmark $\pi_e^\star$, so that eqs. (4) and (5) directly entail that Tobin's q increases.

Regarding the sentiment traders, assume that their benchmark π_e^s coincides with π_e^m. Hence the indicator of optimism z_a in (15) is positive. On account of eq. (16) a herding towards optimism sets in, which is then reinforced by the positive excess capital gains $(\hat{p}_e - \pi_e^s)$ in (15). As a result, the demand d^s of the sentiment traders is rising. On the other hand, the resulting rise of $\hat{p}_e$ (and therefore q) diminishes the fundamentalist demand d^f. Eventually, q will reach q^f, when d^f becomes zero, and then rise above q^f, from when on d^f turns negative.

Total demand of the speculative traders is still positive but less than

[11] The relationship between the transition probability approach, the discrete choice approach and the present somewhat 'smoothed' discrete choice approach is considered in detail in Franke (2013).

before, which implies that the rise of the stock prices decelerates. This also weakens the positive feedback in the sentiment dynamics. Even though the sentiment index a may increase further, from the remark on (16) we know that this motion will eventually be reversed. From here on, the sentiment traders reduce their demand. Sooner or later not only d^f but total demand as well will be negative, $\hat{p}_e$ gets lower than $\pi_e^m = \pi_e^\star$, and Tobin's q starts falling; see (4) and (5).

What has just been described is a global stabilization mechanism, which certainly works in both directions. After the turning point of, first, the sentiment index a and, subsequently, Tobin's q, the stock market may converge to an equilibrium position, adjustments that could take place in a monotonic or a cyclical manner. Alternatively, there may be an overshooting of the equilibrium in every new round so that persistent fluctuations come into being. Lastly, it might even be the case that near a (positive) turning point of a the negative d^f and the positive d^s balance. In a neighbourhood of this configuration a situation can arise where simultaneously $\dot{a} = 0$ and $\dot{q} = 0$. Besides a balanced equilibrium regarding the sentiment traders, this would establish another stock market equilibrium with a distinct majority of optimists. More definite statements, however, about which of these possibilities prevails—and under what conditions—require a mathematical analysis.

3.2 Stability and instability in the two-dimensional system

By virtue of the specifications in Section 2 it is possible to reduce the stock market dynamics to a system of two differential equations. To this end, substitute the demand terms in the price equation (8) by (12) and (14), which yields

$$\hat{p}_e \;=\; \pi_e^m \;+\; \beta_e\,[\,\beta_s a + \beta_f(q^f - q)\,] \tag{17}$$

The derivative of Tobin's q is then readily obtained from (4) and (5),

$$\dot{q} \;=\; q\,\Big\{\,\pi_e^m - \pi_e^\star \;+\; \beta_e\,\Big[\,\beta_s a + \beta_f(q^f - q)\,\Big]\,\Big\} \tag{18}$$

The second dynamic variable is the sentiment index a. Since only (a, q) enter the stock price changes in (17), the indicator of optimism z_a in (15) is a function of the same two variables, and this equally holds true for the right-hand side of (16) governing the motions of a. Equations (16) and (18) are thus seen to constitute the two-dimensional system that we aimed at.

It may also be noted that nothing changes if in the expression γz_a in (16) the coefficient γ is multiplied by some positive number and ϕ_a, ϕ_p in z_a in (15) by its reciprocal. Hence we can put $\gamma = 1$ without loss of generalization. The same observation can be made for the coefficients β_e, β_s and β_f in (18), so that β_e can be likewise put equal to unity. With this type of scaling, our stock market model finally reads as follows:

$$\begin{aligned} \dot{a} &= \nu\,\{\,\tanh\,[z_a(a,q)] \;-\; a\,\} \\ \dot{q} &= q\,[\,(\pi_e^m - \pi_e^\star) \,+\, \beta_s a \,+\, \beta_f(q^f - q)\,] \\ z_a(a,q) &= \phi_a a \,+\, \phi_p\,[\,(\pi_e^m - \pi_e^s) \,+\, \beta_s a \,+\, \beta_f(q^f - q)\,] \end{aligned} \tag{19}$$

It is quite convenient for the analysis that the system exhibits only one essential nonlinearity, namely, the hyperbolic tangent in the equation for the sentiment index. As it has already been mentioned, this feature is of the same kind as in Lux (1995) except that there the expression in curly brackets is multiplied by a positive cosh term. Stock price changes in the Lux paper are governed by a linear combination of the sentiment index and the price misalignment, while here owing to the growth rate formulation of Tobin's q the relationship is quasi-linear.

Nevertheless, if one assumes identical benchmark rates $\pi_e^m = \pi_e^s = \pi_e^\star$, system (19) and the Lux system produce the same isoclines for (in the present notation) $\dot{a}$ and $\dot{q}$, that is, the same geometry in the phase plane would prevail. Because of the slight modifications, however, the Jacobian matrices in some of the equilibrium points will not be precisely identical. In particular, parameter configurations might exist where an equilibrium is stable in one system and unstable in the other, although from a practical point of view such a parameter region would be rather small.

The two isoclines $\dot{q} = 0$ and $\dot{a} = 0$ in the (a,q) phase plane are indeed the basis for an analysis of the stock market dynamics. Owing to the case distinctions for the three benchmark rates of stock price inflation, the formulation of the mathematical propositions that can be derived from them will appear somewhat technical. Later, we will therefore add a numerical study since its phase plots, and the changes they are undergoing when one of the parameters is varied, will be more vivid.

Considering the two isoclines, the $\dot{q}$-isocline is easily seen to be given by a linearly increasing function $q = q_{ICq}(a)$. Also the relationship $\dot{a} = 0$ can be explicitly solved for q, although the corresponding function $q = q_{ICa}(a)$ is more involved.[12] It can be shown that the two isoclines intersect at a balanced sentiment $a = a^o = 0$, which together with a suitable value q^o of Tobin's q constitutes an equilibrium point of system (19).

The function $q = q_{ICa}(a)$ is nonlinear and always tends to $+\infty$ as $a \to -1$ and to $-\infty$ as $a \to +1$. It can be everywhere strictly decreasing, in which case (a^o, q^o) is the only point that the two isoclines have in common. On the other hand, $q = q_{ICa}(a)$ can be an increasing function over some medium range of a. Here it depends on the relative slope of the two isoclines at

[12]For this and the other mathematical statements to follow, see Appendix D.

(a^o, q^o) whether it remains the only equilibrium or whether in addition two outer equilibria emerge. By virtue of the asymptotic behaviour of $q = q_{ICa}(a)$ for $a \to \pm 1$, the latter happens if at (a^o, q^o) this function is steeper than the other isocline $q = q_{ICq}(a)$.[13] Maintaining the equality of the two benchmark rates π_e^s and $\pi_e^\star$, there is a simple condition for the uniqueness of the balanced equilibrium (a^o, q^o), and a symmetry statement for the case of multiple equilibria.

Proposition 1. *Suppose that the stock price inflation benchmark of the sentiment traders π_e^s is equal to the objective benchmark $\pi_e^\star$ from (5). Then the following holds.*

(a) The point $E^o = (a^o, q^o)$ with $a^o = 0$ and $q^o = q^f + (\pi_e^m - \pi_e^\star)/\beta_f$ constitutes an equilibrium of system (19).

(b) The equilibrium E^o is unique if and only if the sentiment traders' herding parameter ϕ_a is less than or equal to unity.

(c) If on the other hand $\phi_a > 1$, two (and only two) additional equilibria $E^1 = (a^1, q^1)$ and $E^2 = (a^2, q^2)$ emerge. They are symmetric around E^o, that is, $a^1 < a^o < a^2$, $q^1 < q^o < q^2$, and $a^2 = -a^1$, $q^2 - q^o = q^o - q^1$.

According to the remark on eq. (13), $q^\star = 1$ appears to be a natural equilibrium value of Tobin's q. The first part of the proposition shows that, except for a fluke, it only constitutes a point of rest of the stock market dynamics if the fundamentalists share this view, i.e. if $q^\star = q^f$, and if additionally the market maker adopts the objective benchmark for stock price inflation, i.e. if $\pi_e^m = \pi_e^\star$. Clearly, this observation has no counterpart in the Lux model.

Dropping the assumption on the stock price inflation benchmarks in Proposition 1 and admitting different values of π_e^m, π_e^s, $\pi_e^\star$ leaves the shape of the $\dot{a} = 0$ and $\dot{q} = 0$ isoclines unaltered. As shown by eqs. (A3), (A2) in Appendix D, they are only shifted upward or downward in the (a, q)-phase plane. Hence, as long as the differences between the benchmark rates are not excessively large, the statements about the existence and location of the equilibrium points remain qualitatively the same. In addition, the next proposition tells us how the sentiment index in an equilibrium E^o is affected by the changes in π^s.

Proposition 2. *Suppose that the difference between π_e^s and $\pi_e^\star$ is not too large. Then the following holds.*

(a) Again, a unique equilibrium $E^o = (a^o, q^o)$ exists if and only if $\phi_a \leq 1$, where $a^o > 0$ if $\pi_e^s < \pi_e^\star$ and $a^o < 0$ if $\pi_e^s > \pi_e^\star$.

[13] Examples of these cases are given in Figures 1 and 2 below.

(b) *If $\phi_a > 1$, three equilibria E^o, E^1, E^2 exist with $a^1 < a^o < a^2$ and $q^1 < q^o < q^2$. Here, for $E^o = (a^o, q^o)$, $a^o < 0$ if $\pi_e^s < \pi_e^\star$ and $a^o > 0$ if $\pi_e^s > \pi_e^\star$.*

(c) *In both cases, the equilibrium value of Tobin's q at E^o is given by $q^o = q^f + [(\pi_e^m - \pi_e^\star) + \beta_s a^o]/\beta_f$.*

Not only the number of equilibria but also their stability properties are crucially dependent on the sentiment traders' herding parameter ϕ_a. It can render E^o stable and unstable, and in the latter case E^o can be repelling or a saddle point. Even the outer equilibria can be both unstable or stable, or one may be stable and the other unstable. In the present limited setting it is thus also possible that the stock market gets stuck in a self-stabilizing bubble, or eventually such a bubble bursts.

If E^o is unique and repelling, the market does not diverge but will persistently oscillate around E^o. Mainly responsible for this pleasant property is the nonlinearity in the sentiment dynamics, which produces a suitable shape of the $\dot{a} = 0$ isocline that allows us to apply the Poncaré-Bendixson Theorem in a mathematical proof. A limit cycle could furthermore exist in the presence of three equilibria. The conditions for stability and instability are the least involved if we concentrate on the strictly symmetric case where $a^o = 0$ at E^o. Clearly, the mathematical statements are maintained when $a^o \neq 0$, only the conditions would be somewhat distorted.

Proposition 3. *Suppose $\pi_e^s = \pi_e^\star$, so that $a^o = 0$ in $E^o = (a^o, q^o)$, and put $A = \phi_p \beta_s - \beta_f q^o/\nu$. Then the following holds.*

(a) *If $\phi_a < 1 - \max\{0, A\}$, the unique equilibrium E^o is (at least) locally asymptotically stable. The inequality $\phi_a < 1 - \phi_p \beta_s$ is a sufficient condition for its global stability.*

(b) *If $A > 0$ and $1 - A < \phi_a < 1$, the equilibrium E^o is repelling and every trajectory outside of E^o converges towards a periodic orbit. At the critical value $\phi_a = 1 - A$, the system undergoes a Hopf bifurcation.*

(c) *The equilibrium E^o is a saddle point if $\phi_a > 1$. The two outer equilibria E^1, E^2 may or may not be locally asymptotically stable, or E^2 may be stable but not E^1. If both E^1 and E^2 are unstable, a locally stable limit cycle exists; otherwise such a closed orbit may or may not exist.*

Regarding the periodic orbit in Proposition 3b, the mathematical analysis is not powerful enough to tell us whether or not it is unique. Likewise, there are no handy conditions for the stability or instability of the outer equilibria, and no conditions at all about the existence of one or several global orbits around one of them, or both of them. Deeper insights into these dynamic properties can be obtained by a numerical investigation, which will be the subject of the next section.

4 A numerical study

It is useful for a numerical analysis to start out from a benchmark scenario. To this end, the parameter values in Table 1 are proposed. The underlying time unit is one year, which affects the interpretation of the coefficients $\nu, \pi_e^m, \pi_e^s, \pi_e^\star$.[14] The scenario was chosen such that (a) there is a unique equilibrium $E^o = (a^o, q^o)$ with $a^o = 0, q^o = q^f = 1$, and (b) the composite coefficient A results as $A = 0.8333$, which with $1 - A < \phi_a < 1$ and Proposition 3b implies that E^o is repelling and all (nondegenerate) motion converges to a closed orbit. As a matter of fact, Figure 1 illustrates that convergence occurs towards a unique limit cycle. Its period is moreover exactly one year and hence considerably shorter than an ordinary business cycle in the real sector.[15] The upper and lower bounds between which the two state variables oscillate are (roughly) $a = 0 \pm 0.60$ and $q = 1 \pm 0.08$. The mechanisms driving such a cycle have already been sketched in Section 3.1.

In the remainder of this section we study how the dynamics is affected by *ceteris paribus* variations of the central herding parameter ϕ_a. They will give rise to a number of different topologies, brought about by a sequence of local and also global bifurcations (regarding equilibrium points and closed orbits, respectively). At the end, we return to the benchmark scenario and introduce an exogenous regular business cycle motion in the real sector, which will shift the stock market up and down with a period of about eight years.

To begin with the variations of the herding parameter, we know from Proposition 3b that the qualitative behaviour of Figure 1 is preserved as long as $\phi_a < 1$. The simulations assure us that this also includes the uniqueness of the limit cycle. Quantitatively, as ϕ_a rises up to 0.99, its period increases to roughly 1.6 years and the amplitudes to $a = 0 \pm 0.88$ and $q = 1 \pm 0.14$. Figure 2, then, summarizes the most important findings as we let ϕ_a increase above unity.

The effect of an increasing ϕ_a is that the $\dot{a} = 0$ isocline in Figure 2 becomes steeper and steeper in E^o. Eventually, it has a steeper slope than the straight line $\dot{q} = 0$. This happens at $\phi_a = 1.00$. By virtue of the nonlinear shape of $\dot{a} = 0$, two (and only two) outer equilibria E^1 and E^2 come into being when $\phi_a > 1.00$. E^o itself turns from a repelling equilibrium into a saddle point. Mathematically, it may therefore be said that the bifurcation

[14] The specific values of $\pi_e^m, \pi_e^s, \pi_e^\star$ play no role (cancel out in (19)) as long as these three benchmark rates are equal. Unequal values would shift the isoclines $\dot{a} = 0$ and/or $\dot{q} = 0$ and thus their points of intersection. Quantitatively, however, these effects would be completely negligible; see eqs (A2) and (A3) in Appendix D, given that $\beta_f = 5$.

[15] This seasonal pattern was motivated by the trading adage, "sell in May and go away" (in order to get back in November).

Table 1: Benchmark scenario of system (19).

	Notation	Values	Description
Sentiment traders:			
	ϕ_a	0.38	herding parameter in z_a, eq. (15)
	ϕ_p	1.25	reaction to price changes in (15)
	ν	12	flexibility parameter in (16)
	β_s	1	reaction coefficient in (14)
Fundamentalists:			
	q^f	1	subjective Tobin's q in (10)
	β_f	5	reaction coefficient in (12)
Stock price inflation:			
	π_e^m	0.02	market maker's benchmark in (8)
	π_e^s	0.02	sentiment traders' bench. in (15)
	$\pi_e^\star$	0.02	objective benchmark rate in (5)

Note: The corresponding composite parameter A from Proposition 3 is $A = 0.8333$, hence $1 > \phi_a > \phi_a^H = (1-A) = 0.1667$, the Hopf bifurcation value of the herding parameter, and E^o is repelling.

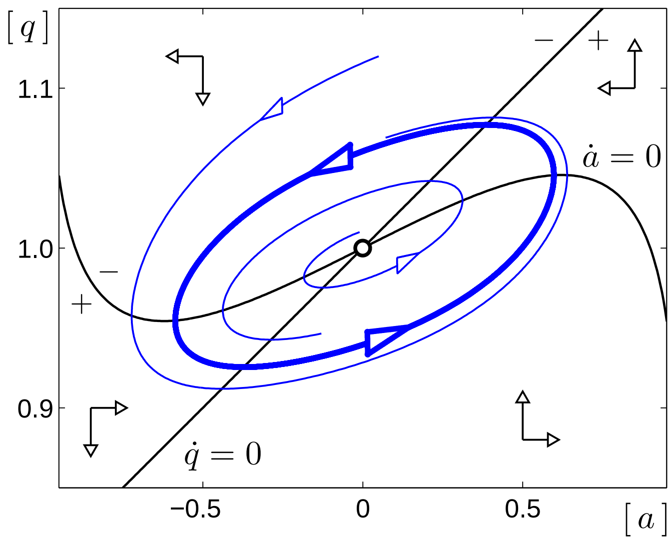
occurring at $\phi_a = 1.00$ is an inverted, or unstable, pitchfork bifurcation.[16] Table 2 below provides a succinct summary of this and the other bifurcations to come.

Initially, as ϕ_a is sufficiently close to 1.00, the outer equilibria E^1 and E^2 are unstable. With the Poincaré-Bendixson theorem it can again be concluded that all (non-degenerate) trajectories converge to a periodic orbit. Figure 2a demonstrates that this orbit is unique, too. It has arisen from the limit cycle in Figure 1, which has gradually widened and now (necessarily) encircles all three equilibria.

The value of $\phi_a = 1.1752$ in Figure 2a is shortly below the critical value $\phi_a = 1.1753$ at which the optimistic bubble equilibrium E^2 (but not E^1)

[16] In an ordinary (supercritical) pitchfork bifurcation in $\mathbb{R}^2$, both eigen-values are initially negative and one of them becomes positive, when simultaneously two outer equilibria appear which are stable (Strogatz, 1994, pp. 246ff). In our case, both eigen-values are initially (real and) positive and one of them becomes negative, while the two new equilibria are unstable (which is not a 'subcritical' pitchfork bifurcation).

Figure 1: Unique equilibrium and convergence towards a unique limit cycle.



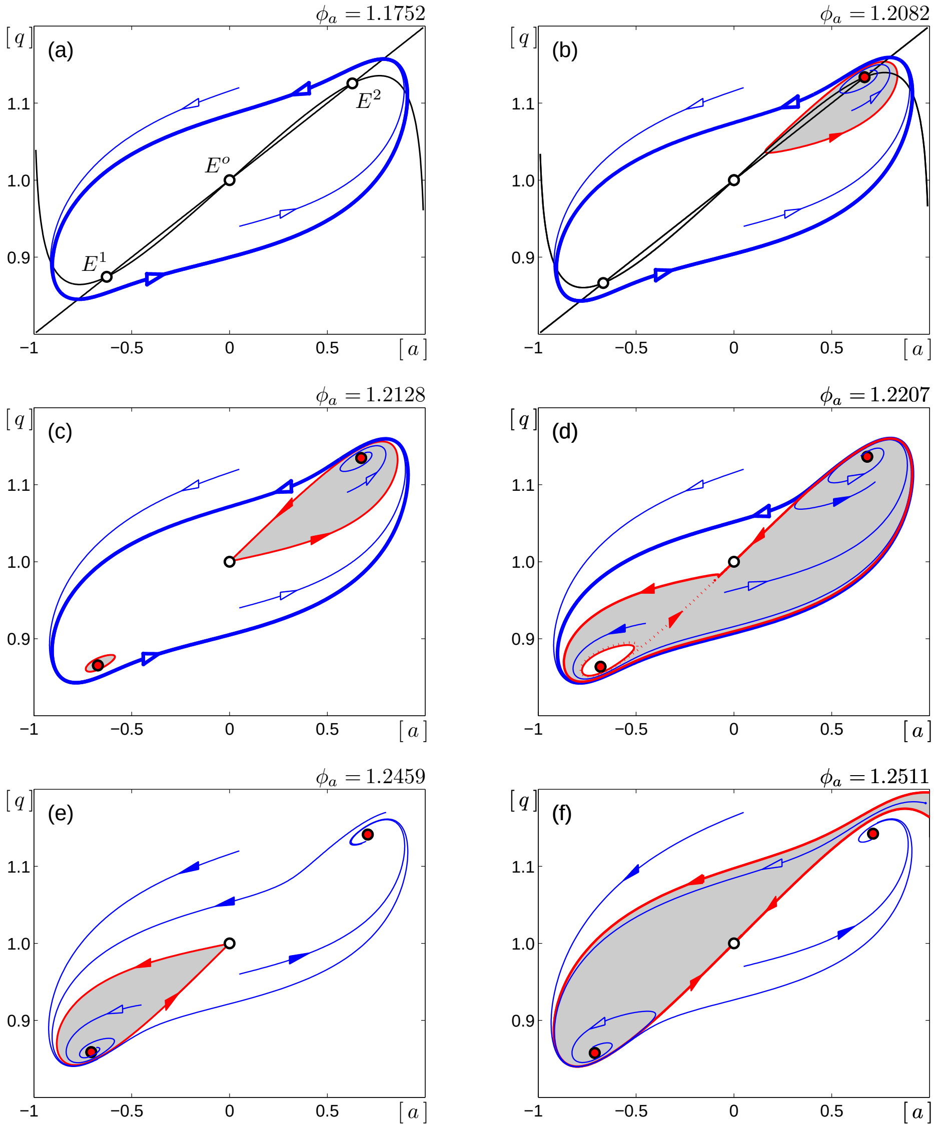
becomes stable. As it was repelling before, this bifurcation is of the Hopf type and E^2 is approached in a cyclical manner. For $\phi_a = 1.2082$, the shaded area in Figure 2b represents the basin of attraction of E^2. It is fenced off by a closed orbit which is repelling from the inside and the outside, and which has emerged from the (subcritical) Hopf bifurcation in E^2. The big limit cycle has not been affected by this local structural change.

Another Hopf bifurcation occurs for the pessimistic equilibrium in E^1 at $\phi_x = 1.2084$. A small basin of attraction around E^1 for a somewhat higher value is shown in Figure 2c, in the lower-left corner (the isoclines are here omitted in order not to overload the plot).

Regarding the increasingly larger basin of attraction of E^2 as the herding parameter rises, it must eventually touch the inner equilibrium point E^o. Here, at $\phi_a = 1.2128$ in Figure 2c, the unstable closed orbit around E^2 becomes a homoclinic orbit. This means, the unstable saddle path that springs from E^o in north-east direction bends around E^2 from below and then returns to E^o as a stable saddle path. A further increase of ϕ_a lets the closed orbit (or the pseudo-closed orbit in the latter case) disappear; it can no longer grow since the saddle point E^o cannot be contained in the interior of the basin of attraction of another equilibrium. The homoclinic bifurcation is therefore of a global type.

Let us next consider in Figure 2d what happens for $\phi_a > 1.2128$ when then big limit cycle is still present. A trajectory starting in its interior either

Figure 2: The impact of a varying herding parameter ϕ_a.



Note: Filled (empty) dots indicate stable (unstable) equilibria. The isoclines in the last four panels are omitted (they look essentially the same as in the first two panels).

converges to it, or to the lower equilibrium E^1 (when starting in the 'eye' around E^1, or to the optimistic equilibrium E^2 (when starting in the shaded area). The dotted line in the south-west part of the diagram is the lower part of the stable manifold of E^o. If we follow it backwards, it converges towards the closed orbit around E^1; after all, it must come from somewhere. This saddle path is also kind of a separatrix: starting south-east of it, the market steers directly in the direction of E^2; starting north-west of it, it first takes a course around the pessimistic bubble equilibrium E^1.

The boundary of the basin of attraction of E^2 is now given by the upper branch of the stable manifold of E^o, the one that converges to E^o from the north-west. If we follow it backwards, we may say that it 'sprang' from point A (which could be shifted arbitrarily close to the lower stable saddle path).

It should be noted that the saddle path starting in A leaves some space to the big limit cycle (which in Figure 2d is so narrow that it is no longer visible). As already observed, rising values of ϕ_a increase the basin of attraction of E^2. At the same time, however, the big limit cycle does not stretch out correspondingly. As a consequence, the two orbits must eventually collide, upon which the big cycle disappears. This is what happens at $\phi_a = 1.2209$, a global event that may be called a cyclic fold bifurcation (Strogatz, 1994, p. 261). Perhaps it may be referred to as a semi-cyclic fold since one of the two colliding trajectories is not a closed orbit proper.

Figure 2e for the higher parameter $\phi_a = 1.2459$ shows two things. First, as just described, the big limit cycle has disappeared. Second, the closed orbit surrounding the basin of attraction of E^1 has become a homoclinic orbit, analogous to the situation for E^2 in Figure 2c.

For the values of ϕ_a beyond this homoclinic bifurcation there are no more closed orbits and the phase plane is divided into two adjacent regions from where the market converges to one of the bubble equilibria, which continue to be locally stable. Figure 2f for $\phi_a = 1.2511$ illustrates that nevertheless these regions will look quite different. Generally, without further information about reasonable starting points, the pessimist equilibrium E^1 has a smaller basin of attraction than its optimistic counterpart E^2. It is also no exception that a trajectory converging to E^2 first has to move around E^1, even very closely so.

By increasing ϕ_a above 1.2511, the global shape of the two basins of attraction can still change in (mathematically) interesting ways. We sketch this in Figure 4 in Appendix E. To conclude our investigation of the different dynamic scenarios in the phase plane, the sequence of bifurcations induced by the *ceteris paribus* changes of the herding parameter ϕ_a is summarized in Table 2 together with the different regimes to which they give rise.

At the end of our numerical investigation we subject the stock market to regular oscillations in the real sector. The feedback channel is given by Tobin fundamental value q^f as the fundamentalists perceive it; see eq. (10).

Table 2: Different regimes of (19) under variations ϕ_a.

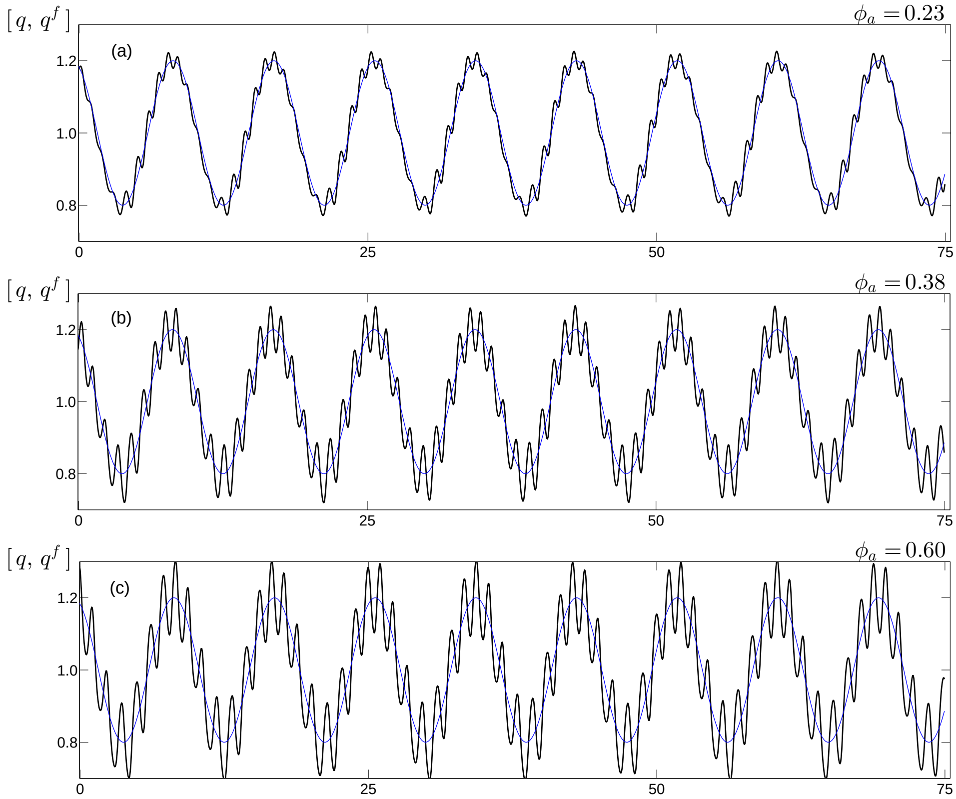
ϕ_a	bifurcation	E^o	E^1	E^2	limit cycles
0.0000					
		stable	—	—	—
0.1667	Hopf in E^o				
		repelling	—	—	unique and stable
1.0000	inverted pitchfork				
		saddle	unstable	unstable	unique and stable
1.1753	Hopf in E^2				
		saddle	unstable	stable	one stable, one unstable around E^2
1.2084	Hopf in E^1				
		saddle	stable	stable	one stable, two unstable ar. E^1, E^2
1.2127	homoclinic				
		saddle	stable	stable	one stable, one unstable around E^1
1.2209	semi-cyclic fold				
		saddle	stable	stable	one unstable around E^1
1.2459	homoclinic				
		saddle	stable	stable	—
2.0000					

While, besides the fundamentalist benchmark rate of stock price inflation π_e^f, q^f is composed of several key variables in the real sector varying over the business cycle (though with phase shifts), it will be convenient here to postulate that it is directly q^f that follows a regular motion; a sine wave with a period of 8.17 years and an amplitude $q^f = 1.00 \pm 0.20$, let us say.[17] We are interested to see how these ups and downs affect the stock market, i.e. Tobin's q in the first instance (after an initial transient time interval is discarded).

Three different values of the herding parameter ϕ_a are considered: the one from the benchmark scenario and one higher, one lower. The top panel of Figure 3 shows the case $\phi_a = 0.23$. This value renders the amplitude of q in the autonomous case so small that now the stock market closely follows the exogenous sine wave of q^f; the intermediate cycles of q of less than one year are rather insignificant.

[17]The 8.17 years was the result of a calibration of a business cycle model that incorporates a general business climate formally similar to (16) into the New Macroeconomic Consensus.

Figure 3: The stock market under a regular oscillation of q^f.



Note: The regular long cycle is a sine wave of q^f with a period of 8.17 years, the short cycles represent the market dynamics of Tobin's q.

Underlying the second panel in Figure 3 is the benchmark value $\phi_a = 0.38$, which gives rise to a larger amplitude. The stock market has thus more life of its own, although every year at another scale and the distance between a trough of q and the subsequent peak depends on the stage of the business cycle, i.e. in particular, whether q^f is presently rising or falling. Since the period of the business cycle is not an integer multiple of the one-year stock market cycle, the motion of q does not repeat itself from one business cycle to another. From a practical point of view, however, it seems that q exhibits a periodicity of five business cycles.

It will come as no surprise that a further increase of ϕ_a to 0.60 in the bottom panel of Figure 3 yields even wider stock market fluctuations. The influence of q^f is nevertheless still visible. On the other hand, here we see no obvious pattern that would reproduce itself over time, at least not within the

75 years of the diagram. Near the lower turning points of the business cycle q^f there are sometimes one and sometimes two stock market cycles, but, on the face of it, all of them with different trough values. In this sense the sine wave business cycle and the intrinsic stock market dynamics generate some form of "chaos", albeit a fairly weak one. Our perspective is whether or not these irregularities may become stronger when the motion of q^f derives from an endogenous business cycle of the real sector, and when q from the stock market (reasonably in a smoother fashion) feeds back on aggregate goods demand.

5 Conclusion

This paper makes a contribution in several respects. First of all, it advances a speculation dynamics on the stock market that is compatible with growth and inflation in the real sector, where the link between the two spheres is provided by Tobin's q. Second, treating the variables in the real sector as given, the seminal two-dimensional Lux (1995) model with its stationary background can be (essentially) recovered from the new stock market as a special case. Apart from that, a new type of agents enter the scene, i.e. absorbers or providers of equities newly supplied to, or withdrawn from, the market. For simplicity, they can remain in the background, while in extensions of the model with a finer market structure they may reappear again. Third, the speculative agents may additionally differ in the benchmark rates of stock price inflation that they apply, which could render the value of Tobin's q in a (balanced) equilibrium different from unity.

As a fourth point, it is also shown (in the appendix) that the dynamic law governing the evolution of a sentiment index can be derived in a straightforward manner, without having to invoke the apparatus of statistical mechanics as in Lux (1995). Fifth but not least, depending on a herding parameter the model can generate persistent cyclical behaviour, in the absence as well as in the presence of an optimistic and a pessimistic bubble equilibrium. Generally, *ceteris paribus* variations of this coefficient can give rise to a sequence of very different local and global bifurcations.

The main motivation for proposing this stock market model was our interest to combine it with a low-dimensional endogenous business cycle model of the real sector, where Tobin's q (directly and indirectly) acts in both directions such that the distinct cycle frequencies in the two sectors are basically maintained (and do not tend to synchronize). The exercise of our exogenous oscillations of a real-sector variable gave us some reasons to expect that the financial-real interaction of the two cycle mechanisms has some potential for complex dynamics ('chaos'), even in a deterministic setting and in continuous time.

Appendix

A. A simple rule to determine the equity growth rate

Consider the finance identity (1). In case of a positive growth of equities, $\dot{E} > 0$, the retention rate will *ceteris paribus* be lower if the new equities can be sold at higher prices. Theoretically, it might even be negative, which means that the dividend payments $(1-\sigma_f)\, r\, pK$ would be defrayed from the profits as well as from part of the receipts from selling the equities, either because the current equity prices are so favourable or the growth rate g_e is so high. However, the latter does not appear to be a policy that could be maintained over a longer period of time. On the other hand, it is obvious that if firms decide to buy back equities, $\dot{E} < 0$, they will have to retain more profits than given by the ratio g/r in (3) in order to finance a given volume of investment. An extreme policy where $-g_e$ is so high that σ_f would exceed unity is most likely to be accompanied by a cutting down of investment, i.e. a suitable reduction of I in eq. (1) to restore $\sigma_f \geq 0$.

Sensible values for a positive growth rate $g_e = \dot{E}/E$ can be deduced from the following argument. Suppose that as a benchmark, firms would like to finance a fixed fraction χ of their investment by equities, $p_e\dot{E} = \chi\, pI$. However, this specification would make their equity growth rate dependent on current stock prices, $\hat{E} = \chi\, g \,/\, q$ (which results from $q\hat{E} = (p_eE/pK)\,(\dot{E}/E) = p_e\dot{E}/pK = \chi\, pI/pK = \chi\, g$). Since firms will not want to subject their equity policy to the volatilities of the stock market, they may replace the actual value of q with a constant level q^n that they consider to be a 'normal' value of Tobin's q. Correspondingly, firms may determine their equity growth rate as

$$g_e \;=\; \chi\, g/q^n$$

B. Derivation of the reduced-form price equation (8)

Let the short-term fundamentalists be identified by a superscript *'fst'*. As indicated in the main text, they are indifferent between buying and selling equities (and correspondingly perhaps between selling and buying government bonds) if the current rate of return on equities is equal to the risk-adjusted interest rate,

$$r_e \;=\; \frac{[\,1-\sigma_f(q)\,]\, r}{q} \;+\; \hat{p}_e \;=\; i+\xi$$

where ξ is the positive risk premium. Using (6), equality prevails if the instantaneous capital gains $\hat{p}_e$ coincide with the rate

$$\pi_e^{fst} \;=\; \pi_e^{fst}(q) \;:=\; i \,+\, \xi \,-\, \frac{r-g}{q} \,-\, g_e$$

In this situation the excess demand d^{fst} of the short-term fundamentalists is just sufficient to absorb $\dot{E}$ from the firms (if $\dot{E} > 0$) or to provide the desired shares $\dot{E}$ to them (if $\dot{E} < 0$). Higher (lower) capital gains than the benchmark π_e^{fst} induce a higher (lower) demand d^{fst}.

The rate $\pi_e^{fst}(q)$ can be relatively volatile and the short-term fundamentalists may not wish to follow each and every change of q. We therefore assume that they adopt a constant benchmark $\bar{\pi}_e^{fst}$. This value may, for example, be obtained from considering some 'normal' value q^n in the expression for $\pi_e^{fst}(q)$, i.e. $\bar{\pi}_e^{fst} = \pi_e^{fst}(q^n)$. In addition, let us specify the excess demand of the short-term fundamentalists as a linear relationship. On the whole, this gives us

$$d^{fst} \;=\; [\,1 + \eta\,(\hat{p}_e - \bar{\pi}_e^{fst})\,]\;\dot{E}$$

where η is a positive reaction coefficient. Now, total excess demand on the market is $d^{fst} + d^f + d^s - \dot{E}$. Plugging this into (7) yields

$$\hat{p}_e \;=\; \tilde{\pi}_e^m \;+\; \tilde{\beta}_e \Big[\,\eta\,(\hat{p}_e - \bar{\pi}_e^{fst})\,\frac{\dot{E}}{E} \;+\; \frac{d^f + d^s}{E}\,\Big]$$

To be exact, underlying $\hat{p}_e$ on the left-hand side is the forward derivative, whereas $\hat{p}_e$ on the right-hand side represents the backward derivative. Technically, however, this does not matter and we can solve the equation for $\hat{p}_e$, where we suppose that the three coefficients $\tilde{\beta}_e$, η, g_e are so small that their product is less than unity. In this way we end up with

$$\hat{p}_e \;=\; \frac{\tilde{\pi}_e^m \,-\, \tilde{\beta}_e\,\eta\,g_e\,\bar{\pi}_e^{fst}}{1 - \tilde{\beta}_e\,\eta\,g_e} \;+\; \frac{\tilde{\beta}_e}{1 - \tilde{\beta}_e\,\eta\,g_e}\,\frac{d^f + d^s}{E} \qquad \text{(A1)}$$

To obtain eq. (8) it only remains to denote the first fraction π_e^m and the second fraction β_e (which is also positive). Note that if, not implausibly, the short-term fundamentalists use the same benchmark for stock price inflation as the market maker, $\bar{\pi}_e^{fst} = \tilde{\pi}_e^m$, the first term in the equation becomes $\tilde{\pi}_e^m$ (and we could omit the tilde).

C. The law of motion for the sentiment index a

If a sentiment trader reconsiders his current attitude at time t, his new decision is governed by the discrete choice probabilities $P^+(z_{a,t})$ and $P^-(z_{a,t})$ mentioned in the text. In particular, they are independent of his previous attitude. A single agent, however, makes this reconsideration only infrequently. Let ν be the fixed probability *per unit of time* for such a reconsideration (which is the same for all sentiment traders). In a discrete-time setting with adjustment period Δt, an agent's probability of operating a random mechanism for P^+ and P^- between t and $t+\Delta t$ is correspondingly $\Delta t\,\nu$, the complementary probability for unconditionally sticking to his present attitude being $(1-\Delta t\,\nu)$.

Ar the macroscopic level, the population shares of optimists and pessimist in the next period $t+\Delta t$, $a^+_{t+\Delta t}$ and $a^-_{t+\Delta t}$, result like

$$\begin{aligned} a^+_{t+\Delta t} &= (1-\Delta t\,\nu)\; a^+_t \;+\; \Delta t\,\nu\, P^+(z_{a,t}) \;=\; a^+_t \;+\; \Delta t\,\nu\,[\,P^+(z_{a,t}) \;-\; a^+_t\,] \\ a^-_{t+\Delta t} &= (1-\Delta t\,\nu)\; a^-_t \;+\; \Delta t\,\nu\, P^-(z_{a,t}) \;=\; a^-_t \;+\; \Delta t\,\nu\,[\,P^-(z_{a,t}) \;-\; a^-_t\,] \end{aligned}$$

Letting the adjustment period shrink to zero, $\Delta t \to 0$, and taking the specification of $P^\pm$ into account, the continuous-time formulation reads,

$$\begin{aligned} \dot a^+ &= \nu\,\Big[\,\frac{\exp(\gamma z_a)}{\exp(\gamma z_a)+\exp(-\gamma z_a)} \;-\; a^+\,\Big] \\ \dot a^- &= \nu\,\Big[\,\frac{\exp(-\gamma z_a)}{\exp(\gamma z_a)+\exp(-\gamma z_a)} \;-\; a^-\,\Big] \end{aligned}$$

and the difference between the two derivatives is

$$\dot a \;=\; \nu\,\Big[\,\frac{\exp(\gamma z_a)-\exp(-\gamma z_a)}{\exp(\gamma z_a)+\exp(-\gamma z_a)} \;-\; a\,\Big]$$

It remains to recall the definition of the hyperbolic sine, cosine and tangent, $\sinh(x) = [\exp(x)-\exp(-x)]/2$, $\cosh(x) = [\exp(x)+\exp(-x)]/2$ and $\tanh = \sinh/\cosh$. The fraction in the last differential equation is thus seen to be equal to $\tanh(\gamma z_a)$. This completes the proof of eq. (16).

Regarding the numerical interpretation of the coefficient ν, two examples may suffice: $\nu = 0.5$ and $\nu = 12$ with respect to an underlying time unit of one year. The first number means the agents reconsider their attitude with a probability of 1/2 per year, or on average every two years. In the second case, ν cannot be directly interpreted as a probability. We reread it as a reconsideration probability of, say, $12/24 = 1/2$ over a time interval of 1/24 years. Accordingly, a reconsideration takes place every $2\times(1/24) = 1/12$ years, or every month.

D. Proof of the mathematical statements

It is easily seen in (19) that under $\pi^s_e = \pi^\star_e$ (and therefore $\pi^m_e - \pi^s_e = \pi^m_e - \pi^\star_e$), $a = a^o = 0$ and $q = q^o$ imply $\dot q = 0$, $z_a(a,q) = 0$, and $\dot a = 0$. Hence E^o in Proposition 1a constitutes an equilibrium without further assumptions.

Let us then continue with the derivation of the two isoclines and their shape. Solving the equality $\dot q = 0$ in (19) yields the straight line

$$q \;=\; q_{ICq}(a) \;:=\; q^f \;+\; (\pi^m_e - \pi^\star_e)\,/\,\beta_f \;+\; \frac{\beta_s}{\beta_f}\,a \qquad \text{(A2)}$$

The equality $\dot{a} = 0$ prevails if $\tanh\{\phi_a a + \phi_p[(\pi_e^m - \pi_e^s) + \beta_s a + \beta_f(q^f - q)]\} = a$. Applying the inverse function $\mathrm{arctanh}(\cdot)$ to both sides of this equation, using the identity $\mathrm{arctanh}(a) = (1/2) \cdot \ln[(1+a)/(1-a)]$, and solving the resulting equation for q, we obtain

$$
\begin{aligned}
q \;=\; q_{ICa}(a) \;&:=\; q^f \;+\; (\pi_e^m - \pi_e^s)/\beta_f \;+\; B_1\, a \;-\; B_2 \,\ln\left[\frac{1+a}{1-a}\right] \qquad \text{(A3)}\\
\text{where}\quad B_1 &= (\phi_a + \phi_p\,\beta_s)\,/\,\phi_p\,\beta_f\,, \quad B_2 = 1\,/\,2\,\phi_p\,\beta_f
\end{aligned}
$$

Clearly, the function tends to $-\infty$ as a approaches 1, and to $+\infty$ as the sentiment index approaches -1 from the right. Its derivative is given by

$$
\frac{d\,q_{ICa}(a)}{d\,a} \;=\; B_1 \;-\; \frac{2\,B_2}{1-a^2} \qquad \text{(A4)}
$$

From this expression it can be inferred that the isocline $\dot{a} = 0$ is everywhere strictly decreasing if the composite parameter B_1 falls short of $2B_2$, which is tantamount to $\phi_a + \phi_p\,\beta_s < 1$. Otherwise the isocline has a positive slope over an intermediate range of a, though it still decreases for a closer to the ± 1 boundaries. Equating the derivative (A4) to zero and solving it for a, it is seen that in this case the isocline has exactly one local minimum (maximum) at a negative (positive) value of the sentiment index.

In this way the two isoclines have two (and only two) additional points of intersection E^1, E^2 if (and only if) at E^o the $\dot{a} = 0$ isocline is steeper than the $\dot{q} = 0$ isocline, that is, if (and only if) $B_1 - 2_2 > \beta_s/\beta_f$, a condition that is equivalent to $\phi_a > 1$. The symmetry property of E^1 and E^2 is obvious from the skew-symmetry of $\dot{q} = 0$ and $\dot{a} = 0$ (the latter follows from (A4)). This observation completes the proof of Proposition 1.

Turning to the proof of Proposition 2, note that the isoclines (A3) and (A2) have been derived without any assumption on $\pi_e^m, \pi_e^s, \pi_e^\star$. Thus also the general problem of determining the equilibrium values of a can be reduced to a single equation by equating the right-hand sides of (A3) and (A2). This gives us

$$
F(\pi_e^s, a) \;:=\; \phi_p\,(\pi_e^\star - \pi_e^s) + \phi_a\, a - \frac{1}{2}\,\ln\left[\frac{1+a}{1-a}\right] \;=\; 0
$$

Since according to (A4) the derivative of the last term in F with respect to a is $1/(1-a^2) \geq 1$, a sufficient (though not necessary condition) for a unique solution of this equation is $\phi_a \leq 1$. Fixing $\pi_e^\star$, the equilibrium value a^o can be viewed as a function of π_e^s. Application of the Implicit Function Theorem then yields

$$
\frac{d\,a^o}{d\,\pi_e^s} \;=\; \frac{-\partial F/\partial \pi_e^s}{\partial F/\partial a} \;=\; \frac{\phi_p}{\partial F/\partial a}
$$

The denominator is negative if $\phi_a < 1$, so that a^o decreases with π_e^s in this case. On the other hand, at least when evaluated at $\pi_e^s = \pi_e^\star$, which by virtue of Proposition 1a implies $a^o = 0$, the denominator is positive under $\phi_a > 1$, in which case a^o increases with π_e^s. Lastly, the equilibrium value q^o of Tobin's q is obvious from $\dot{q} = 0$ once a^o is determined.

For the proof of Proposition 3 we note the following. Not for the local but for a global stability analysis it is useful to define $\tilde{q} = \ln q$ and rewrite system (19) as follows:

$$\begin{aligned} \dot{a} &= \nu \{ \tanh [z_a(a,\tilde{q})] \; - \; a \} \\ \dot{\tilde{q}} &= (\pi_e^m - \pi_e^\star) + \beta_s a + \beta_f [q^f - \exp(\tilde{q})] \\ z_a(a,\tilde{q}) &= \phi_a a + \phi_p \{ (\pi_e^m - \pi_e^s) + \beta_s a + \beta_f [q^f - \exp(\tilde{q})] \} \end{aligned} \tag{A5}$$

With $\tanh' = 1/\cosh^2$, the Jacobian matrix is then generally given by

$$J = J(a,\tilde{q}) \quad = \quad \begin{bmatrix} \nu \left\{ \dfrac{\phi_a + \phi_p \beta_s}{\cosh^2[z_a(a,\tilde{q})]} - 1 \right\} & \dfrac{-\nu\, \phi_p \beta_f\, q}{\cosh^2[z_a(a,\tilde{q})]} \\ \beta_s & -\beta_f\, q \end{bmatrix}$$

where $q = \exp(\tilde{q})$. It may be noted that for the Jacobian of system (19), the lower diagonal entry would be $j_{22} = \partial \dot{q}/\partial q = -\beta_f\, q$ plus another term that is zero in equilibrium but else different from zero, which makes an unambiguous sign assessment for the trace more difficult or even impossible. Working with (A5) avoids this complication and, defining $C(a,\tilde{q}) := \cosh^2[z_a(a,\tilde{q})]$, the trace and the determinant result like

$$\begin{aligned} \operatorname{trace} J(a,\tilde{q}) &= \nu\, [(\phi_a + \phi_p \beta_s)/C(a,\tilde{q}) - 1] - \beta_f\, q \\ \det J(a,\tilde{q}) &= \nu\, \beta_f\, q\, [1 - \phi_a/C(a,\tilde{q})] \end{aligned}$$

The assumption ensuring $a^o = 0$ in E^o implies $z_a(a^o,\tilde{q}^o) = 0$ and $C(a^o,\tilde{q}^o) = 1$. Hence $\det J(a^o,\tilde{q}^o) > 0$ if and only if $\phi_a < 1$, and $\operatorname{trace} J(a^o,\tilde{q}^o) < 0$ if and only if $\phi_a < 1 - A$. This establishes the conditions for the local stability, instability and saddle point behaviour of E^o in Proposition 3. Since upon a *ceteris paribus* rise of ϕ_a the loss of stability occurs at some value $\phi_a^H < 1$ where the determinant is still positive, ϕ_a^H constitutes a Hopf bifurcation.

Furthermore, because of $C(a,\tilde{q}) \geq 1$ we can be sure that under $\phi_a < 1 - \phi_p \beta_s$, entry j_{11} and therefore the trace of $J(a,\tilde{q})$ is negative for all $(a,\tilde{q})$ with $-1 \leq a \leq 1$, while the determinant is always positive. Moreover, none of the entries changes its sign. From Olech's Theorem it can thus be concluded that E^o is also globally stable (cf. Gandolfo, 1997, pp. 354f).

Convergence towards a periodic orbit in case E^o is repelling can be proved by means of the Poincaré-Bendixson Theorem. Returning to the formulation of eq. (19), it suffices to realize that there is a rectangle in the

(a, q)-plane such that every trajectory starting there remains there, and every other trajectory eventually enters it. The rectangle is given by the two vertical boundaries $a = \pm 1$, and the two horizontal lines $q = q^L$ and $q = q^H$, where q^L and q^H are the two values of Tobin's q where the linearly increasing $\dot{q} = 0$ isocline hits the $a = -1$ and $a = 1$ axis, respectively (a diagram constructed in this way can be found in Franke, 2012, Figure 1). If q^L happens to be less than zero, the lower horizontal line is simply given by $q = 0$.

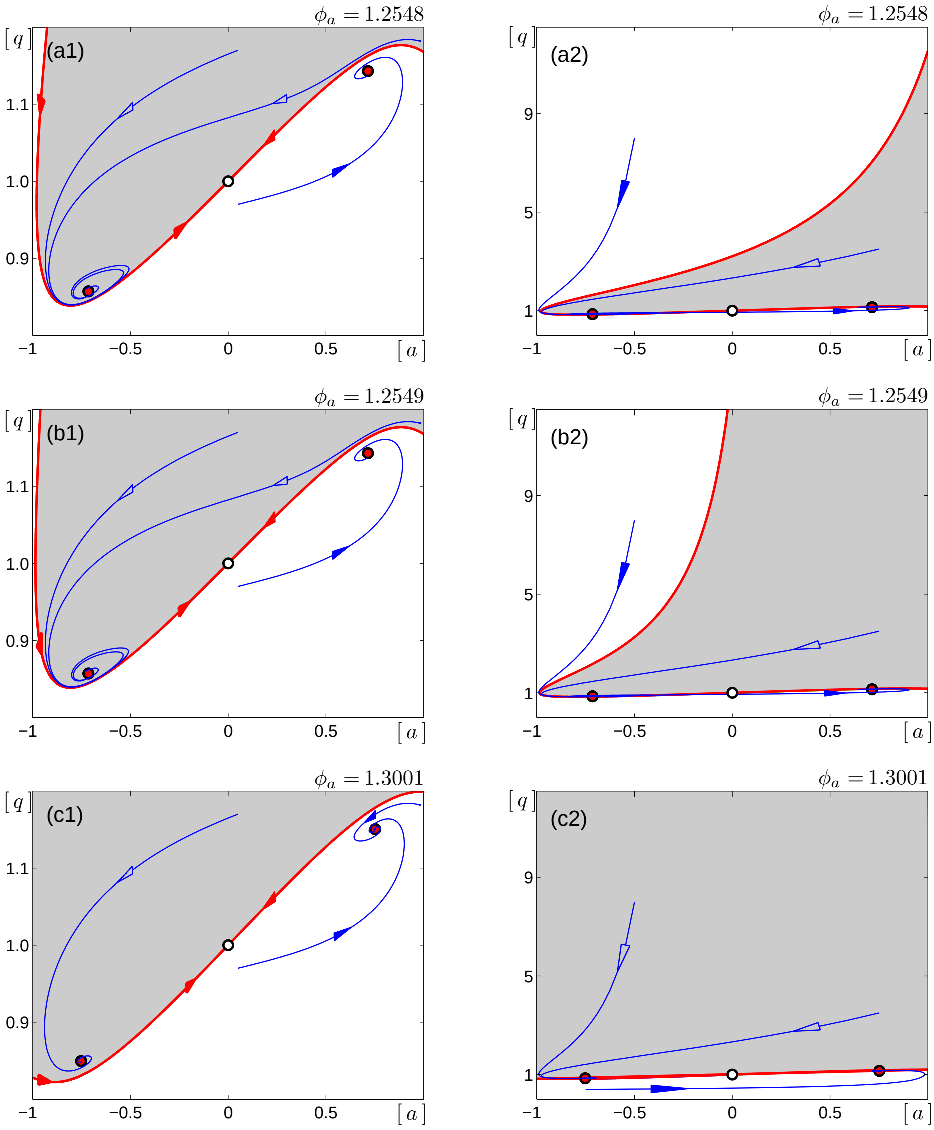
Note that the conclusion holds as well if E^o is a saddle, the closure of the basins of attraction of the other equilibria do not fill out the entire plane, and the trajectory does not start on a stable saddle path.

E. The phase plane for high values of the herding parameter

For situations where herding is so strong that there are no more prospects for cyclical behaviour, the basins of attraction of the two bubble equilibria E^1 and E^2 are adjacent and separated by the stable manifold of the interior, balanced equilibrium E^o. Figure 4 illustrates that this separatrix may nevertheless exhibit quite different features, although at such high values of Tobin's q that they are only of mathematical interest. Regarding q, panel (a1) of Figure 4 for $\phi_a = 1.2548$ shows that the basin of attraction of E^1, BA(E^1), is bounded from below, whereas BA(E^2) has neither an upper bound nor a strictly positive lower bound. Panel (a2) clarifies the latter for extremely high values of q, and reveals that BA(E^1) is also bounded from above.

Panels 4 (b1) and (b2) demonstrate that a slight increase of ϕ_a to 1.2549 causes BA(E^1) to be unbounded from above as well, without interfering with the unboundedness of BA(E^2) except that its horizontal width along the a-axis is reduced. A further increase of ϕ_a lets this part of BA(E^2) shrink progressively more, until it eventually disappears. This has happened in panels 4 (c1), (c2) at $\phi_a = 1.3001$, where basin BA(E^2) is bounded from above and BA(E^1) bounded from below.

Figure 4: Basins of attraction of the two bubble equilibria as the herding parameter ϕ_a gets (very) large.



Note: The shaded areas indicate the basin of attraction of the pessimistic bubble equilibrium E^1

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