Technical Report

Models of Asset Price Dynamics From Beja–Goldman to Brock–Hommes: A Synopsis of Specification Details

Reiner Franke*

Department of Economics
University of Kiel
Kiel, Germany

March 2008

Abstract

The report gives an overview of small-scale asset pricing models. Following the tradition of the seminal Beja–Goldman model, they are typically populated by two archetypal groups of speculative traders, namely, fundamentalists and chartists (also called trend chasers or technical traders). The population shares may be fixed or, in the discrete choice framework introduced by Brock–Hommes, may endogenously evolve in a dynamic process of evolutionary fitness. There is a great variety of such models in the literature which often differ only in some, as it seems, minor details. In order to organize this material, a number of features concerning prices, expectations, demand and the evolutionary process are distinguished, for each of which several alternative specifications have been formulated over the past one or two decades. The report then concentrates on three main lines of research which, for short, can be associated with the universities of Amsterdam, Leuven and Sydney. For each line five to seven models are chosen from the literature and classified under the abovementioned features, where the assignments are conveniently summarized in a tabular synopsis.

*Mail to franke@iksf.uni-bremen.de.

Financial support from EU STREP ComplexMarkets, contract number 516446, is gratefully acknowledged.

Contents

1 Introduction

In theoretical financial economics it is now widely acknowledged that a paradigm shift has taken place from the representative agent with his or her rational expectations towards a behavioural approach, in which markets are populated by heterogeneous and boundedly rational agents who use rule-of-thumb strategies. As a consequence, a great variety of such models have been put forward over the past one or two decades, which at first sight seem to provide a wilderness of formulations that one easily lose track of.

There are nevertheless certain modelling principles that, more or less clearly discernible, can often be reencountered in these models. This begins with the two archetypal groups of chartists (also called trend followers or technical traders) and fundamentalists. The general intuition that the demand of fundamentalists tends to be stabilizing and that of chartists tends to be destabilizing has been most influentially substantiated in the seminal model by Beja and Goldman (1980). While chartist (though not so much fundamentalist) strategies have been modified or generalized over the following years, the original mechanisms from Beja–Goldman still appear to shine through in many other models and remain helpful to an understanding of what is going on in different episodes of a model-generated price history.

Another feature are the population shares (or market fractions) of the (mostly) two groups of agents, which are to be thought of as being fixed in the Beja–Goldman model. An important line of research has dropped this assumption and advanced dynamic evolutionary processes where the population shares endogenously vary over time according to some measure of fitness, i.e., differential profits that the corresponding strategies have earned in the past. The main framework that has been employed for this kind of modelling is the so-called discrete choice model, which has been introduced into the financial markets context by Brock and Hommes.

If the interest is restricted to asset pricing models with some more or less remote connections to the concepts of Beja–Goldman and Brock–Hommes and one plunges into that literature, one receives a somewhat contradictory overall impression. On the one hand, there is still a great multitude of models. On the other hand, at least within the work of some selected groups of authors certain model building blocks tend to reappear, either in exactly the same or in a slightly modified form. So their models seem to be "quite similar" to each other, though a specific model will usually claim originality with respect to the others. It is here where the present report sets in. Its main purpose is to provide a guideline through all the specification details of models with some (possibly loose) elements from Beja–Goldman and/or Brock–Hommes. It does this by distinguishing a number of modules like total supply of the asset, formation of the market price, specification of chartist demand, etc., and then for each of these modules collects several alternative specifications from the literature. In this way a classification scheme is set up that can serve to give an overview of the basic features of a model, in particular, when they are contrasted with the features of other models that bear some resemblance to it.

In the latter sense, the classification will also be applied to the literature. Specifically, we identify three research centers, or even "schools", in the cities of Amsterdam, Leuven, and Sydney. For each of them five to seven papers are chosen and subjected to the classification. The presentation of these results in a synoptical table will then be the upshot of the report.

The organization of the material is as follows. The next two section discuss the single model building blocks and the different specifications to be found in the literature. Section 2 deals with the issues concerning market price formation, the agents' price expectations, and a coverage of their risk considerations in the formulation of demand, while Sections 3 discusses the different aspects of the evolutionary process that determines the time path of the market fractions of fundamentalists and chartists. Section 4 reviews the classification in a compact form and presents its application to the literature from the three research centers just mentioned.

2 Prices, expectations and demand

2.1 Total supply

Agents trade in a market with one risky and one risk-free asset. While the latter is in infinite supply, supply of the risky asset z_{st} in period t is limited. Occasionally it is treated as a positive constant $\bar{z}_s$, but most often z_{st} is assumed to be zero over the entire time horizon. In the context of foreign exchange markets, there is also the scenario of a central bank policy leaning against the wind by varying the current account. Using the acronym TS as a mnemonic of 'total supply' of the risky asset, we list the three cases as follows:

$$z_{st} = 0 \qquad \text{for all } t \tag{TS-0}$$

$$z_{st} = \bar{z}_s \neq 0 \qquad \text{for all } t \tag{TS-1}$$

$$z_{st} \quad \text{determined by central bank policy} \tag{TS-2}$$

In some stochastic model versions (with Walrasian market clearing as in (MP-1) below), (TS-0) allows z_{st} to be normally distributed around zero (e.g., Chiarella, He and Wang, 2006a, p. 5).

2.2 Individual demand

Let $R > 1$ be the fixed gross rate of return of the risk-free asset, p_t the price of the risky asset in period t and y_{t+1} the (stochastic) dividend per share over the same period, which is paid out at the beginning of the next period. The asset's excess returns ξ_{t+1} earned between t and $t+1$ are then given by

$$\xi_{t+1} \;=\; p_{t+1} + y_{t+1} - R p_t \tag{1}$$

Agents are mean-variance maximizers with a uniform risk aversion coefficient a.[1] Agents of type h have conditional expectations $E_{ht}(\xi_{t+1})$ about excess returns, with conditional variance $\sigma^2_{ht} = \mathrm{Var}_{ht}(\xi_{t+1})$. If an agent of this type buys z_h shares, his expected risk-adjusted profits are $z_h\, E_{ht}(\xi_{t+1}) - \frac{a}{2}\,\mathrm{Var}_{ht}(z_h\, \xi_{t+1})$. He therefore has to solve the optimization problem,

$$z_h\, E_{ht}(\xi_{t+1}) \;-\; \frac{a\,\sigma^2_{ht}}{2}\, z_h^2 \;=\; \max_{z_h}! \tag{2}$$

from which his demand z_{ht} for period t is determined as

$$z_{ht} \;=\; \frac{E_{ht}(\xi_{t+1})}{a\,\sigma^2_{ht}} \tag{3}$$

It is well-known that the same result is obtained from maximizing the expected utility of wealth, if the utility function $U = U(W)$ exhibits constant absolute risk aversion (CARA) and the returns are supposed to be normally distributed. Typically a negative exponential function $U(W) = -\exp(-aW)$ is used in this case. A most convenient property of (3) is its independence of the agents' current wealth position, even though W may enter the formulation of the optimization problem.[2]

2.3 The basic structure of price expectations

The dividend process follows the normal distribution $y_{t+1} \sim N(\bar{y}, \sigma^2_y)$. Both types of agents have correct expectations on dividends and so forecast $E_{ht}(y_{t+1}) = E_t(y_{t+1}) = \bar{y}$. In this way the expectations of excess returns boil down to the agent-specific expectations of prices,

$$E_{ht}(\xi_{t+1}) \;=\; E_{ht}(p_{t+1}) + \bar{y} - R p_t \tag{4}$$

[1] In a few works (He and Li, 2007, and Dieci et al., 2005), fundamentalists and chartists are allowed to differ in their risk aversion a. In the numerical simulations, however, this feature is not exploited and the coefficients are uniform again. Since a affects the agents' demand in a proportionate way (see (3) in a moment), the same effect in total demand can be achieved by a variation in the populations shares of the two groups.

[2] An optimization under a utility function with constant relative risk aversion (CRRA) yields the same expression (3), if there z_{ht} does not denote the agent's absolute demand for the asset but the proportion of his wealth that he decides to invest. The details are, for example, spelled out in the appendix of Chiarella and He (2001).

These expectations are decomposed into a base price p_t^b, which is the same for all agents, and a function B_h which describes the specific way in which agents of type h form their beliefs. Generally, we thus have

$$E_{ht}(p_{t+1}) \;=\; p_t^b + B_h \tag{5}$$

The entries in the function B_h are in the first instance the prices that have been most recently observed. Fundamentalists and some chartists will relate them to the fundamental value, and a few models let their chartists also adopt an additional predetermined variable to forecast $E_{ht}(p_{t+1})$.

Before discussing the belief functions B_h, consider the base price p_t^b, which may be the price of the present period p_t, the price of the previous period p_{t-1}, or the fundamental price p^f. If the latter is variable at all, agents may or may not be assumed to know p_{t+1}^f. However, since fundamental prices are governed by an exogenous univariate stochastic process (see below), the dating of p^f does not matter here, so that in the interest of a uniform notation a fundamental price in the price expectations may be referred to as p_t^f.

On the other hand, models differ in the agents' information set, that is, whether or not it contains the price of the current period t. This depends on the assumption on price formation. As a rule, agents in period t are supposed to know p_t if market disequilibrium is admitted and prices are reset at the beginning of each period by a market maker to reduce the volume of excess demand. Alternatively, agents only know p_{t-1} as the most recent price if Walrasian temporary equilibrium is assumed (the details of these cases are given in the next subsection).[3] Since many features of the models that we want to present can have their place in both scenarios, we may generally introduce a lag parameter τ that takes the value 0 or 1 according to which scenario applies:

$$\begin{array}{l} p_{t-\tau} \text{ is the most recent price in the agents' information set, where} \\ \quad \tau = 1 \quad \text{if price } p_t \text{ is to clear the market} \\ \quad \tau = 0 \quad \text{if price changes are determined by (nonzero) excess demand} \end{array} \tag{6}$$

Throughout the paper, use of the parameter τ will be confined to refer to this information set, which is underlying the price expectations $E_{ht}(p_{t+1})$. If no qualification is added, a specification invoking τ will be meaningful for either value 0 or 1.

[3] In this case, eq. (4) might be more scrupulously reformulated as $E_{ht}(\xi_{t+1}) = E_{ht}(p_{t+1}) + \bar{y} - R\,p_{t,k}$, where, to remain within the tâtonnement parable, $p_{t,k}$ is the price cried out by the Walrasian auctioneer in the k-th round to find the market clearing price p_t for period t. It is then simpler for the agents (and/or the model builder) if they do not have to update their expectations $E_{ht}(p_{t+1})$ at every new quotation $p_{t,k}$.

To make the arguments explicit that may enter the belief functions B_h, we denote the price record in the information set as $p_{[t-\tau,\cdots]} = (p_{t-\tau}, p_{t-\tau-1}, \ldots)$ (with as many lags back into the past as necessary in the model at hand). Other arguments than p^f_t and $p_{[t-\tau,\cdots]}$ may here be indicated by a simple dot. We can thus return to the basic structure of price expectations, where it will be useful to distinguish the two cases whether the base price p^b_t in (5) is given by the fundamental value or by the most recent price in the agents' information set,

$$E_{ht}(p_{t+1}) = p_{t-\tau} + B_h(p^f_t, p_{[t-\tau,\cdots]}, \cdot)\,, \qquad \text{i.e. } p^b_t = p_{t-\tau} \qquad \text{(BP-1)}$$
$$E_{ht}(p_{t+1}) = p^f_t + B_h(p^f_t, p_{[t-\tau,\cdots]}, \cdot)\,, \qquad \text{i.e. } p^b_t = p^f_t \qquad \text{(BP-2)}$$

The acronym BP is to be mnemonic of the 'base price' underlying the price expectations.[4]

2.4 The determination of market prices

Total demand for the risky asset is given by $n_{ft}\, z_{ft} + n_{ct}\, z_{ct}$, where n_{ft} and n_{ct} are the populations shares of fundamentalists and chartists, respectively ($n_{ft} + n_{ft} = 1$). If the agents' information set does not contain p_t and price expectations are based on the price of the previous period, $p^b_t = p_{t-\tau}$ with $\tau = 1$ as in (BP-1), this demand can be equated to total supply z_{st} and easily solved for the Walrasian temporary equilibrium price p_t. Using (3), (4) and letting MP stand for 'market price', the outcome is,

$$p_t = \frac{1}{R}\Big[\, p_{t-1} + \bar{y} + \frac{n_{ft}\, B_f(p^f_t, p_{[t-1,\cdots]}, \cdot)}{\tilde{n}_t\, \sigma^2_{ft}} + \frac{n_{ct}\, B_c(p^f_t, p_{[t-1,\cdots]}, \cdot)}{\tilde{n}_t\, \sigma^2_{ct}} - \frac{a\, z_{st}}{\tilde{n}_t} \Big]$$
$$\tilde{n}_t := \frac{n_{ft}}{\sigma^2_{ft}} + \frac{n_{ct}}{\sigma^2_{ct}} \qquad \text{(MP-1)}$$

The equation is considerably simplified if $z_{st} = 0$ and the conditional variances are uniform, $\sigma^2_{ft} = \sigma^2_{ct} = \sigma^2_t$. The variances cancel out in this case and (MP-1) becomes

$$p_t = \frac{1}{R}\Big[\, p_{t-1} + \bar{y} + n_{ft}\, B_f(p^f_t, p_{[t-1,\cdots]}, \cdot) + n_{ct}\, B_c(p^f_t, p_{[t-1,\cdots]}, \cdot) \Big] \qquad \text{(MP-1a)}$$

In addition, one may specify a suitable functional form for the fundamental beliefs (see below) and postulate a constant fundamental price $p^f_t = p^f = \bar{y}/(R-1)$, so that expected dividends can be written as $\bar{y} = (R-1)\, p^f$. These systems can then be entirely formulated in terms of deviations from the fundamental value, that is, all price expressions with lag τ are of the form

[4] In introductory remarks to their models, some papers state that the agents have price expectations of the general form $E_{ht}(p_{t+1}) = E_{ht}(p^f_{t+1}) + f_h(p_{t-1}, \ldots, p_{t-L})$. This, however, contradicts their explicit specification of chartists expectations in a later section as $E_{ht}(p_{t+1}) = p_{t-1} + g\,(p_{t-1} - p_{t-2})$, which corresponds to a base price $p^b_t = p_{t-1}$ (Gaunersdorfer, 2000b, pp. 11, 14; Hommes, 2001, pp. 154, 159; Gaunersdorfer, Hommes and Wagner, 2006, p. 6).

$x_{t-\tau} := p_{t-\tau} - p^f$. This is a standard procedure in many models of, especially, the Amsterdam group (for example, Gaunersdorfer, Hommes and Wagner, 2003, 2006).

The same kind of reasoning applies if the underlying price expectations are specified as (BP-2) with $\tau = 1$. In any case, under the simplifications given the equilibrium price results like

$$p_t \;=\; p_t^f \;+\; \frac{1}{R}\,\Big[n_{ft}\, B_f(p_t^f, p_{[t-1,\cdots]},\,\cdot\,) + n_{ct}\, B_c(p_t^f, p_{[t-1,\cdots]},\,\cdot\,) \Big] \qquad \text{(MP-1b)}$$

In principle, a Walrasian equilibrium price could be quite as well determined if price expectations are dated differently from (PE-1) and include the period-t price p_t. If the belief functions B_f and B_c are linear in p_t, the calculations to solve the equilibrium condition for p_t would not be too complicated, either. In the BH framework this device is nevertheless not utilized, mainly perhaps for fear that it might introduce unduly diverging tendencies into the model. Besides, one can easily do without it.

The alternative to (MP-1) is to admit disequilibrium on the asset market and introduce a market maker. Though still being highly stylized, this figure is said to bring the analysis closer to the functioning of real markets than the fictitious Walrasian auctioneer (Chiarella and He, 2003, p. 504). In order to maintain an orderly market and to make a profit by doing so (Day, 1997, p. 9), the market maker sets excess demand to zero at the end of each trading period by taking an off-setting long or short position, and announces the next period price as a function of the original excess demand.[5] In this scenario one has lag $\tau = 0$ in (BP-1) and (BP-2). Depending on whether the base price in these price expectations is p_t of p_t^f the market price adjustments read,

$$p_{t+1} \;=\; p_t \;+\; \mu \Big\{ \sum_{h=f,c} \frac{n_{ht}\,[p_t + \bar{y} + B_h(p_t^f, p_{[t,\cdots]},\,\cdot\,) - Rp_t]}{a\,\sigma_{ht}^2} \Big\} \qquad \text{(MP-2)}$$

$$p_{t+1} \;=\; p_t \;+\; \mu \Big\{ \sum_{h=f,c} \frac{n_{ht}\,[p_t^f + \bar{y} + B_h(p_t^f, p_{[t,\cdots]},\,\cdot\,) - Rp_t]}{a\,\sigma_{ht}^2} \Big\} \qquad \text{(MP-3)}$$

The coefficient $\mu > 0$ measures the speed at which the market maker (therefore the Greek letter μ) adjusts the price. Or conversely, its reciprocal $1/\mu$ can be interpreted to represent market depth: a given transaction can be expected to yield stronger price changes when market depth is low and so there are larger price gaps in the order book. Obviously, (MP-2) and (MP-3) treat μ as a constant. This assumption is in fact so standard that it is hardly ever discussed.[6]

[5] For simplicity, however, neither the possible large positive or negative inventory positions of a market maker are made explicit nor the commission he charges on the transactions. Papers investigating this issue in a still very elementary framework include Gu (1995), Day (1997) and Farmer (2001).

[6] Chiarella and He (2003, p. 508) are explicit in admitting a conceptual problem with a fixed value of μ. They add that μ is best thought of as a market friction, and an aim of their analysis is to understand how this friction affects the market dynamics.

There are occasional hints in the literature that the price impact function, $p_{t+1} - p_t$ as a function of the order size, may display a concave curvature and flatten out as the order size increases.[7] In the formulation above this would mean that the coefficientμ is not fixed but decreases as the absolute values of the single demand components are rising. A straightforward specification to capture this idea is

$$\mu \;=\mu_t \;=\; \frac{\bar{\mu}}{\bar{\mu}_o \,+\, (n_{ct}|z_{ct}| + n_{ft}|z_{ft}|)^{\bar{\mu}_1}} \qquad \text{in (MP-2,3)} \qquad \text{(MP-4)}$$

The bar over the symbols μ emphasizes that these coefficients are constant and distinguishes them from the overall (now variable) adjustment parameter μ_t. It goes without saying that $\bar{\mu} > 0$ and $\bar{\mu}_o$, $\bar{\mu}_1$ are nonnegative. A positive coefficient $\bar{\mu}_o > 0$ may be adopted if one wants to avoid infinitely large values of μ_t as z_{ct} and z_{ct} both converge to zero. For $\bar{\mu}_o = 0$, however, the price increments from combining (MP-4) with (MP-2) or (MP-3) are still a continuous function at the equilibrium $(z_c, z_f) = (0, 0)$, given that $\bar{\mu}_1 < 1$; and the adjustment speeds μ_t would remain bounded if the dynamics never come close to this equilibrium. Trivially, one is back at $\mu = \text{const}$ if $\bar{\mu}_1 = 0$.

2.5 Fundamental values

The fundamental value (FV) of the asset is either supposed to be constant over time or to follow a random walk. In one example we know the random walk postulate is generalized to an ARCH(1)-process. The specifications of p^f_t are therefore:

$$\begin{aligned} p^f_t &= p^f = \text{const} \quad \text{for all } t && \text{(FV-0)} \\ p^f_t &= p^f_{t-1} + \varepsilon_t\,, \qquad \varepsilon_t \sim N(0, \sigma^2_{pf}) && \text{(FV-1)} \\ p^f_t &= p^f_{t-1} + \varepsilon_t\,, \qquad \varepsilon_t = \varepsilon^o_t \sqrt{\alpha_{\varepsilon 0} + \alpha_{\varepsilon 1}\, \varepsilon^2_{t-1}}\,, \qquad \varepsilon^o_t \sim N(0,1) && \text{(FV-2)} \end{aligned}$$

The price variability brought about by a random walk can be more easily assessed if the one-period standard deviation in its formulation relates to the relative return $(p^f_t / p^f_{t-1} - 1)$ of the fundamental value, rather than to its level. This version of (FV-1), to which we assign the same label, reads,

$$p^f_t \;=\; (1 + \varepsilon_t)\, p^f_{t-1}\,, \qquad \varepsilon_t \sim N(0, \sigma^2_{rf}) \qquad \text{(FV-1)}$$

[7]Westerhoff (2004, p. 1006), who also proposes the following specification, gives a brief introduction to recent evidence and counter-evidence of this hypothesis. It seems an evaluation of this discussion has to pay particular attention to the length of the adjustment period that is underlying these investigations.

To get an impression of a reasonable order of magnitude of the standard deviation σ^2_{rf} we can refer to He and Li (2007, pp. 3401, 3404), who assume an annual return volatility of 20%. Thus for a daily market period (and 250 days per year) they get $\sigma_{rf} = 0.20/\sqrt{250} = 0.01265$.

2.6 Fundamentalist beliefs

Concerning the belief functions B_f and B_c in the determination of prices, let us begin with the fundamentalist traders. They regard deviations from fundamental values as temporary phenomena and expect that sooner or later actual prices will return to them. The specification of this idea is rather standardized: the expected price is a weighted average of the fundamental price and the most recent price $p_{t-\tau}$ in the agents' information set ($\tau = 0, 1$),[8]

$$E_{ft}(p_{t+1}) \;=\; \tilde{\phi}\, p^f_t \;+ (1-\tilde{\phi})\, p_{t-\tau} \qquad (0 \le \tilde{\phi} \le 1) \tag{7}$$

In terms of the otherwise convenient belief function B_f and referring to (BP-1) and (BP-2), respectively, this kind of expectation formation can be rewritten as

$$B_f \;=\; B_f(p^f_t, p_{t-\tau}) \;=\; \phi\,(p^f_t \;-\; p_{t-\tau}) \tag{FB-1}$$

The Greek letter ϕ is to be reminiscent of the fundamentalists and FB points to 'fundamentalist beliefs'. The sign of the coefficient ϕ depends on the underlying base price in (BP-1) or (BP-2). If $p^b_t = p_{t-\tau}$, one has $\phi = \tilde{\phi}$ with respect to (7) and ϕ is in the range between 0 and 1. The coefficient thus measures how fast fundamentalists believe prices to move back from $p_{t-\tau}$ in the direction of the fundamental value, or how confident they are in this tendency. On the other hand, in the presence of $p^b_t = p^f_t$ the correspondence to (7) is $\phi = \tilde{\phi} - 1$, and $-1 \le \phi \le 0$.

If a paper introduces fundamentalist expectations directly in the form of (7), we are free to insinuate a base price $p^b_t = p_{t-\tau}$ or $p^b_t = p^f_t$. Of course, one would then use the base price that is underlying the specification of the chartist expectations.

Several papers reduce fundamentalist to the extreme belief $E_{ft}(p_{t+1}) = p^f_t$. Others wish to attribute a minimal stabilization potential to the fundamentalist traders and so work with the opposite polar case $E_{ft}(p_{t+1}) = p_{t-\tau}$. These expectations might be simply called naive, or the traders are viewed as believing in the Efficient Market Hypothesis, according to which prices do not exhibit any systematic pattern but follow a random walk.[9]

[8]Note that if the current fundamental price is known to the agents but not its future values, then under the random walk hypothesis the price p^f_t is the best forecast they can make. Else, as indicated above, p^f_t may denote the fundamental price relevant for the next period.

[9]Which, given their rational expectations about dividends and the fundamental value of the asset, leads one to ask for the motive of these traders to be on the market at all.

A short remark on fundamentalist demand in the Beja-Goldman framework may be added, which is there directly postulated to be proportional to the gap between the fundamental value and the current price. As the approach employs a market maker who quotes the price p_t at the beginning of period t, we have here (for $\alpha_f = \text{const} > 0$)

$$z_{ft} \;=\; \alpha_f\,(p_t^f - p_t) \tag{8}$$

This demand is easily seen to be compatible with the mean-variance optimization result $z_{ft} = E_{ft}(\xi_{t+1})/a\sigma^2_{ft}$ in (3), if we neglect the fact that the Beja-Goldman approach usually refers to log prices. Just suppose $\sigma^2_{ft} = \sigma^2 = \text{const}$, $y_t \equiv 0$ and $R = 1$, so that $E_{ft}(\xi_{t+1}) = E_{ft}(p_{t+1}) - p_t$, and plug in (7) with $\tau = 0$ for the price expectations. The coefficient α_f in (8) is then given by $\alpha_f = \tilde{\phi}/a\sigma^2$.

A slight modification of the belief function (FB-1) takes transaction costs into account. This concept was developed in a context of foreign exchange and goods markets (De Grauwe and Grimaldi, 2006, pp. 5f), but it can also be carried over to stock markets. In a general interpretation of the idea it can be said that prices must be outside an indifference band around the fundamental value if fundamentalists are to have faith that the market will return toward p_t^f (De Grauwe and Grimaldi call this band a transaction cost band). Inside the band, the fundamentalists consider this tendency to be insignificant. Hence in the first case the beliefs (FB-1) are maintained, while in the second case the best what can be forecasted is a zero change. With respect to a given "transaction cost" $C > 0$ and a base price $p_t^b = p_{t-\tau}$ as in (BP-1) being understood, the modified belief function reads,

$$B_f \;=\; B_f(p_t^f, p_{t-\tau}) \;=\; \begin{cases} \phi\,(p_t^f \;-\; p_{t-\tau}) & \text{if } |p_{t-\tau} - p_t^f| > C \\ 0 & \text{else} \end{cases} \tag{FB-2}$$

The implication of the indifference band is that the model (or its deterministic skeleton) will possess a continuum of stationary points. This indeterminacy (if C is large enough) may enrich the deterministic as well as the stochastic dynamics.

2.7 Chartist beliefs

Though the behaviour of chartists could be modelled in a great variety of ways, the specifications that have been developed in the literature are rather limited and partly even extremely simple. A first distinctive feature is whether chartists are assumed to make reference to fundamental values or not. In the first case their base price will typically be $p_t^b = p_t^f$ as noted in (BP-2), in the latter case $p_t^b = p_{t-\tau}$ as in (BP-1) (again, $\tau = 0, 1$ depending on the agents' information set).

One straightforward type of chartists does without fundamentals and believes that prices continue to move further in the direction they have observed over the past few periods. With respect to a rolling sample period T_c, the extrapolation of the recent price changes is described as

$$B_c \;=\; B_c(p_{[t-\tau,\cdots]}) \;=\; \gamma \textstyle\sum_{j=0}^{T_c-1} \alpha_{\gamma j}\,(p_{t-\tau-j} \;-\; p_{t-\tau-j-1}) \qquad (\sum_j \alpha_{\gamma j} = 1) \qquad \text{(CB-1)}$$

(γ as the third letter in the Greek alphabet and in this sense corresponding to the Latin 'c' may remind the reader of the group of chartists, while CB stands for 'chartist beliefs'). It goes without saying that the coefficients $\alpha_{\gamma j}$ are weights between 0 and 1. The degree of trend-following in the rule is measured by a positive extrapolation coefficient $\gamma > 0$. More generally, (CB-1) can also capture contrarian views by assuming $\gamma < 0$; see Chiarella and He (2002, pp. 101, 111ff), Chiarella et al. (2004, pp. 5, 13).

The simplest case of (CB-1) is $T_c = 1$ and $\alpha_{\gamma 1} = 1$. Since it has been used in several articles as the only alternative, we list it as an extra equation,

$$B_c \;=\; B_c(p_{t-\tau}, p_{t-\tau-1}) \;=\; \gamma\,(p_{t-\tau} \;-\; p_{t-\tau-1}) \qquad \text{(CB-1a)}$$

Despite its extreme simplicity, (CB-1a) seems to suffice for many purposes. However, if the trading period is a day and the entire model succeeds in reproducing the stylized fact of insignificant autocorrelation in the raw returns, then these agents would have to be accused of being persistently ignorant about one of the most obvious observations they can make. Or the feedbacks from (CB-1a) produce positive autocorrelation in the returns (with $\gamma > 0$), in which case the model would not match all of the empirical features one may desire.

While maintaining the idea that the beliefs B_c are a "trend", i.e., a summary statistic of the observed price changes, this magnitude can also be treated as a dynamic state variable q_t that gradually changes over time. A most convenient specification in this respect are partial adjustments towards the most recent price change. This concept gives rise to the following belief function:

$$\begin{aligned} q_t &= q_{t-1} \;+\; \alpha_q\,(p_{t-\tau} - p_{t-\tau-1} - q_{t-1}) \qquad (0 < \alpha_q < 1) \\ B_c &= B_c(q_t) \;=\; \gamma\, q_t \end{aligned} \qquad \text{(CB-2)}$$

The case $\alpha_q = 1$ is ruled out since it would lead us back to (CB-1a). (CB-2) has its origin in the Beja-Goldman framework, where the perceived trend directly determines the level of demand by the chartists. That is, with a (positive) proportionality factor α_{ct} which in general may be time-varying, the approach specifies

$$z_{ct} \;=\; \alpha_{ct}\, q_t \qquad (9)$$

The compatibility of (9) with z_{ct} from (3) is analogous to the discussion of (8). The variability of α_{ct} can be captured by a variable conditional variance σ^2_{ct} (to be treated below), so that the parameters in (9) and (CB-2) are related by $\alpha_{ct} = \gamma / a\sigma^2_{ct}$.

Alternatively, z_{ct} may be directly specified as a nonlinear S-shaped function of q_t. Choosing the model's parameters such as to render the equilibrium unstable, this is an easy way to obtain persistent and bounded cyclical behaviour in the large. The strategy was first pursued by Chiarella (1992).

The adjustment of q_t in (CB-2) are formally of the adaptive expectations type. A general remark on the reasonableness of this mechanism is given in Section 2.8 below. For the moment being it suffices to mention that, as it has just been noted, (CB-2) is at least more general than (CB-1a), and that in the formation of a new theoretical framework major criteria for its usefulness are the dynamical properties it may generate.

The trend can also be determined in a slightly more sophisticated way, where the chartists behave as (very) elementary econometricians who estimate the trend as a straight line through the price observations. Doing this over a rolling sample period of length T_c, we have

$$\begin{aligned} q_t &= \text{slope of simple regression of } p_{t-\tau}, p_{t-\tau-1}, \ldots, p_{t-T_c} \text{ against time} \\ B_c &= B_c(q_t) \;=\; \gamma\, q_t \end{aligned} \tag{CB-2a}$$

A side effect of this approach is that, as discussed in Section 2.8, chartists could use the residuals of the regression (CB-2a) for their expectations about the conditional variance σ^2_{ct} in the mean-variance optimization.

The regression approach has been labelled (CB-2a) because this equation and (CB-2) are more closely related to each other than it might seem at first sight. To make this more precise we refer to a very similar estimation of the trend to (CB-2a). Instead of the levels, let it be the first differences $p_\tau - p_{\tau-1}$ that are regressed on time, i.e. on $\beta_o + \beta_1 \tau$ for $\tau = t, t-1, \ldots, t-T_c$ (putting $\tau = 0$; in addition to (CB-2a) the coefficient β_1 would thus permit the agents to measure a possible accelerationist element in the evolution of prices). With estimates $\hat\beta_o$ and $\hat\beta_1$ in period t, the trend is then given by $q_t = E_{ct}(p_{t+1} - p_t) = \hat\beta_o + \hat\beta_1(t+1)$. The point we want to make is that this determination of the trend is well approximated by the formula for q_t in (CB-2) (a proof is given in Franke, 1992). In fact, the adjustment speed α_q is linked to the sample period T_c by the relationship

$$\alpha_q \;=\; 4 \,/\, (T_c + 1) \tag{10}$$

Equation (10) may be also of independent interest, since the time horizon T_c provides an alternative to evaluate the order of magnitude of the coefficient α_q in (CB-2).

While the chartists in (CB-1) and (CB-2) seek to extrapolate the recent price movements, another type of chartists considers the level of these prices and relates it to the fundamental value. On the one hand, they take p_t^f for their base price as in (BP-2), and on the other hand they expect the current deviation of these prices from p_t^f to be maintained in the next period, with a tendency of this gap to widen or to narrow down. With respect to a lag $\tau = 0$ or 1 and a rolling sample period T_c, the specification of this concept is

$$\begin{aligned} m_t &= \sum_{j=0}^{T_c-1} \alpha_{\gamma j}\, p_{t-\tau-j} \\ B_c &= B_c(p_t^f, p_{[t-\tau,\cdots]}) \;=\; \gamma\,(m_t - p_t^f) \end{aligned} \qquad \text{(CB-3)}$$

Clearly, $\gamma < 1$ ($\gamma > 1$) captures the idea of a narrowing (widening) deviation from the fundamental value. Regarding the weighted moving-average process in (CB-3), three types are discussed in the literature (e.g., Chiarella and He, 2003, p. 509): a pure moving average with $\alpha_{\gamma j} = 1/(T_c{+}1)$; a process of finite length with geometrically declining weights; and such a process with infinite memory, that is, $B_c = \gamma\,(p_t^m - p_t^f)$ and the adjustment equation for the moving average is $p_t^m = \alpha_m p_{t-1}^m + (1{-}\alpha_m)(p_t - p_t^f)$ for some α_m between 0 and 1. A degenerate specification in (CB-3) is

$$B_c \;=\; B_c(p_t^f, p_{[t-\tau,\cdots]}) \;=\; \gamma\,(p_{t-\tau} \;-\; p_t^f) \qquad \text{(CB-3a)}$$

If here in addition $\gamma < 1$, the chartists are effectively transformed into the fundamentalists of eq. (7) with $\tilde{\phi} = 1{-}\gamma$. The other type of traders in models assuming (CB-3a) are then hardcore fundamentalists with $E_{ft}(p_{t+1}) = p_t^f$ (Gaunersdorfer, 2000a, pp. 804, 807; Hommes, Huang and Wang, 2005, p. 1047).

Another type of chartists expectations can be designed by reconsidering the fundamental value in (CB-3a). If traders have no precise information about the fundamental value (or have no confidence in it), they may replace p_t^f with a benchmark m_t constructed from the history of observed prices. For the same reason, their base price cannot be the fundamental value, so (BP-1) with $p_t^b = p_{t-\tau}$ will here be understood. Let us first follow Chiarella, He and Hommes (2006, p. 1733) and specify the benchmark m_t as an arithmetic average of past prices over a finite sample period of length T_c, which gives us

$$\begin{aligned} m_t &= \frac{1}{T} \sum_{j=0}^{T_c-1} p_{t-\tau-j} \\ B_c &= B_c(p_{t-\tau}, m_t) \;=\; \gamma\,(p_{t-\tau} - m_t) \end{aligned} \qquad \text{(CB-4)}$$

The belief function is here written down as a linear function of the gap between $p_{t-\tau}$ and m_t. Classifying papers under (CB-4) may, however, also include cases where B_c is a nonlinear function of the gap with a ceiling and a floor. A popular example for such a function is the hyperbolic tangent, $B_c = \tanh[\gamma\,(p_{t-\tau} - m_t)]$ (Chiarella, He and Hommes, 2006, p. 1733).

From a practitioner's point of view, (CB-4) amounts to a simple moving average (MA) trading rule, more sophisticated versions of which are widely used among professional day traders (a representative reference is Taylor and Allen, 1992; see p. 307). This approach is based on the idea that a long-period MA signals an imminent break in trend, or the emergence of a new trend, when the MA is crossed by the current price of the asset (or more generally by a shorter MA, a refinement that is here ignored). Correspondingly, a buy (sell) signal is generated when the price rises above (or falls below) the long-period MA.

In descriptions of the MA trading rule it is often not immediately clear whether a long (short) position at a given investment level is taken when the price crosses the MA line from below (above), and then remains unchanged until the next crossing occurs; or whether alternatively the trader is to buy and sell each day according to the buy/sell signal he receives, where the volume of the trades may be fixed or prescribed by the current gap between the price and the MA. (CB-4) in fact conforms to the latter interpretation. If we put $R = 1$ for simplicity and neglect dividends, then with (BP-1) it is directly seen that the demand z_{ct} for the asset is proportional to $E_{ct}(\xi_{t+1}) = E_{ct}(p_{t+1}) - p_{t-\tau} = \gamma\,(p_{t-\tau} - m_t)$.[10]

The scope for the profitability of technical trading rules and, in particular, MA rules has been intensively investigated in the literature; see Brock et al. (1992), Boswijk et al. (2000), Kwon and Kish (2002), James (2005), to name a few. The broad result that, at least under some (straightforward) refinements, MA rules may indeed turn out to make positive profits in the long-run, establishes the high relevance of (CB-4a), which therefore appears to deserve more attention in future work than it has received so far. One modification that is common practice may also be included in this kind of modelling, namely, the introduction of a band around the MA line to the effect that the buy/sell signals only hold outside the band. In this way "whiplash" signals should be eliminated when the price stays close to the MA and possibly crosses it in both directions within short intervals of time (there are other devices with the same purpose).

Discussions of models employing (CB-4a) may also have an eye on the presence of pronounced trends, which are a conceptually desirable (or even necessary) condition for the meaning-

[10]To tame possible divergent tendencies of the price dynamics in the presence of an—implicitly—constant conditional variance σ^2_{ct}, Chiarella, He and Hommes (2006, p. 1733) introduce a nonlinearity in demand by postulating that z_{ct} is an S-shaped function of the gap $(p_{t-\tau} - m_t)$.

fulness of a MA trading rule. To quote an advice from an Internet tutorial on technical trading (Farley, 2007): “Crossovers [of MA’s] add horsepower to many types of trading strategies. But try to limit their use to trending markets. Moving averages emit false signals during the “negative feedback” of sideways markets. Keep in mind these common indicators measure directional momentum. They lose power in markets with little or no price change.” Or the finding emphasized by James (2005, p. 426) that in price series “which do not trend, there is not much hope of profit regardless of the trend parameters used!”.

To give a positive example, it seems that that the simulations presented in Chiarella, He and Hommes (2006, p. 1747, Fig. 8) do satisfy this requirement, although the issue is not explicitly discussed by the authors.[11] A rule of thumb for MA rules is that “the change [i.e. the trend in one direction] typically needs to last for a period around twice the length of the moving average for you to make money” (Incademy.com, 2007). In practice and in the empirical investigations, the specification of the long period for the MA ranges from 50 to 200 or 250 days. This might also be a useful hint for model simulations (in their, as it appears, most promising stochastic simulations, Chiarella, He and Hommes, 2006, work with a realistic period of 100 days for the MA).

Instead of the simple arithmetic MA, practioners occasionally adopt an exponentially weighted MA. With an infinitely long period, this is also an attractive parsimonious specification for theoretical models. Since an infinite sum of prices $p_{t-\tau-j}$ with geometrically declining weights $(1-\alpha_m)\alpha_m^j$ $(j = 0, 1, 2, \dots)$ is equivalently represented by an adaptive expectations mechanism, (CB-4) becomes in this case

$$\begin{aligned} m_t &= \alpha_m\, m_{t-1} \;+\; (1-\alpha_m)\, p_{t-\tau} \qquad\qquad (0 \le \alpha_m \le 1) \\ B_c &= B_c(p_{t-\tau}, m_t) \;=\; \gamma\,(p_{t-\tau} - m_t) \end{aligned} \tag{CB-4a}$$

The belief function (CB-4a) is, e.g., used in He (2003, p. 7) or He and Li (2007, p. 3402), though the authors do not (any longer) mention its interpretation as a MA trading rule. In fact, from that point of view α_m should not be chosen too small.[12]

[11] The figure also illustrates the lag until a trend reversal is recognized and the rule gives an opposite buy/sell recommendation. However, as another Internet tutorial (Incademy.com, 2007) remarks on this phenomenon, “Better to be on the right side of the market, and leave a bit of profit for others, than try to squeeze every single penny out of the market and end up taking a hit”.

[12] Interestingly, in his setting He (2003, p. 40) finds that values of α_m exceeding some threshold $\alpha_m^\star$ between 0.70 and 0.80 render the fundamental steady state unstable.

2.8 Time-varying conditional variances

In order to avoid premature complications of the model, it seems natural to many authors to leave heterogeneity in the conditional variances $\sigma^2_{ht} = \mathrm{Var}_{ht}(\xi_{t+1})$ aside and, moreover, treat this risk measure as a constant. With CV as a mnemonic of conditional variances, this view can be referred to as

$$\sigma^2_{ft} \;=\; \sigma^2_{ct} \;=\; \sigma^2 \;=\; \text{const} \tag{CV-0}$$

There are nevertheless also several papers that allow for endogenous variations of σ^2_{ft} and/or σ^2_{ct} over time. However, while stating that "this is an important generalization" (as in Gaunersdorfer, 2000a, p. 806, for example), most of these papers are not very explicit about their general motivation. In this respect it might therefore be interesting to mention an early paper by Franke and Sethi (1993, 1998) on asset price dynamics that has anticipated this idea and is very clear about the effects that may thus be generated. The paper employs the Beja-Goldman (1980) model and so models the demand of chartists as being directly proportional to the (positive or negative) expected slope of log prices. The innovation by Franke and Sethi in this framework is the notion that chartists perceive a greater risk the more dispersed the prices have been in the recent past, and that their demand decreases (in modulus) as the risk increases. Clearly, by specifying chartist demand in a mean-variance optimization framework as $z_{ct} = E_{ct}(\xi_{t+1})/a\sigma^2_{ct} = [E_{ct}(p_{t+1}) - p_t]/a\sigma^2_{ct} = \text{expected slope}/a\sigma^2_{ct}$ (neglecting dividends and the risk-free interest rate), the same effect could be achieved by a rising variance in the denominator of the demand term.

The dispersion is measured by the variance v_t of the residuals from the regression (CB-2a) of past log prices on time, and it reduces the original demand of chartists by a factor $1/(1+\alpha\, v_t)$. The reasons for introducing the risk aversion mechanism are twofold: it has a stabilizing effect in the outer regions of the state space, and it has a chaos generating potential. The first point is easily understood by noting that the original deterministic Beja-Goldman model, which is linear, would converge toward the steady state position if the fixed weight of chartists in the market is below a certain threshold value, and the trajectories would diverge in a cyclical manner if the weight is (moderately) above it. Now, the dispersion increases as the prices approach an upper or lower point. Since the multiplicative factor just mentioned can be viewed as corresponding to the weight of chartists, the influence of chartists is weakening and the fundamentalists can eventually stabilize the global dynamics.

On the other hand, price changes become more and more regular and the dispersion d_t decreases as the market approaches the fundamental value again. Hence the unstable chartist regime is re-established and, on the whole, bounded and persistent price fluctuations are generated. In addition it turns out that the system can exhibit complex dynamics where, in particular, the mean values of the peak-to-peak fluctuations are significantly above or below normal over longer but unpredictable periods of time.

Let us then return to the standard BH framework and consider the variance $\mathrm{Var}_{ht}(\xi_{t+1})$. It is here explicitly or implicitly assumed for all agents that $\mathrm{Cov}_{ht}(p_{t+1} - Rp_t, y_{t+1}) = 0$ and $\mathrm{Var}_{ht}(y_{t+1}) = \sigma_y^2 = \text{const}$. The conditional variance is thus the sum of two single variances, $\sigma_{ht}^2 := \mathrm{Var}_{ht}(p_{t+1} - Rp_t + y_{t+1}) = \mathrm{Var}_{ht}(p_{t+1} - Rp_t) + \sigma_y^2$.

A first specification of the variable part of σ_{ht}^2 in the more recent models of BH type assumes homogeneous expectations of fundamentalists and chartists. They are updated as a weighted average of the variance Var_{ht} in the previous period and the most recent contribution to it. The latter is the squared deviation of $p_{t-1} - Rp_{t-2}$ from a moving average value $m_{\xi,t-1}$ of this difference, which is likewise adjusted in a gradual manner. In detail (again for lags $\tau = 0, 1$),

$$\begin{aligned}
\sigma_{ht}^2 &= \mathrm{Var}_{ht}(p_{t+1} - Rp_t) + \sigma_y^2 = v_t + \sigma_y^2 \,, \qquad h = c, f \qquad (\sigma_y^2 = \text{const}) \\
v_t &= \alpha_v \, v_{t-1} + (1-\alpha_v)\,(p_{t-\tau} - Rp_{t-\tau-1} - m_{\xi,t-1})^2 \\
m_{\xi t} &= \alpha_m \, m_{\xi,t-1} + (1-\alpha_m)\,(p_{t-\tau} - Rp_{t-\tau-1}) \qquad (0 \le \alpha_v, \alpha_m \le 1)
\end{aligned} \tag{CV-1}$$

Instead of the lagged moving average $m_{\xi,t-1}$ in the middle row one could, of course, also employ its contemporaneous value $m_{\xi t}$. It seems that the dating in (CV-1) facilitates the stability analysis of the system's equilibrium point(s), though the dynamics of a model should not be seriously affected by this issue.

A second specification of time-varying variances supposes a constant variance for the fundamentalists and uses the historical variance to scale up the fundamental variance for the chartists (Chiarella, He and Wang, 2006, p. 114). In detail, the variations of Var_{ct} derive from the following features. (i) Instead of $p_{t+1} - Rp_t$, the variance of the price p_{t+1} itself is considered. This amounts to the (implicit) assumption that p_{t+1} and p_t are not systematically correlated, or that the agents are not aware of that. (ii) The excess of $\mathrm{Var}_{ct}(p_{t+1})$ over $\mathrm{Var}_{ft}(p_{t+1})$ is represented by a perceived variance v_t (times a coefficient), which carries a different interpretation but is formally determined in a similar way to (CV-1). Taken together, the conditional variances of fundamentalists and chartists are given as follows:

$$
\begin{aligned}
\sigma^2_{ft} &= \mathrm{Var}_{ft}(p_{t+1}) \;+\; \sigma^2_y \;=\; \sigma^2_p + \sigma^2_y && (\sigma^2_p, \sigma^2_y = \text{const}) \\
\sigma^2_{ct} &= \mathrm{Var}_{ct}(p_{t+1}) \;+\; \sigma^2_y \;=\; \sigma^2_p + \alpha_\sigma v_t + \sigma^2_y && \\
v_t &= \alpha_v \, v_{t-1} \;+\; \alpha_v \, (1-\alpha_v) \, (p_{t-\tau} - m_{t-1})^2 && \\
m_t &= \alpha_m \, m_{t-1} \;+\; (1-\alpha_m) \, p_{t-\tau} && (0 \le \alpha_v = \alpha_m < 1)
\end{aligned}
\tag{CV-2}
$$

In He and Li (2007, p. 3402), Dieci et al. (2005, p. 5), chartists have belief functions (CB-3) whose long-run sample mean m_t is the same as m_t in (CB-4). Again, 'mean' is to be understood as the mean of all prices discounted with α_m into the infinite past. In (CV-2) the same principle is assumed to apply to v_t, where, however, care has to be taken of the dating of m in the squared deviations of p from m. In explicit terms, with $\alpha_m = \alpha_v = \alpha$ and lag $\tau = 0$, the variables m_t and v_t in (CV-2) are equivalently given by

$$
m_t \;=\; (1-\alpha) \sum_{j=0}^{\infty} \alpha^j \, p_{t-j} \;, \qquad v_t \;=\; (1-\alpha) \sum_{j=0}^{\infty} \alpha^j \, (p_{t-j} - m_t)^2 \tag{11}
$$

(the different dating of m in the formulae for v_t, m_{t-1} in (CV-2) and m_t in (11), is correct; the proof of the equivalence of v_t in (CV-2) and (11) is quite tedious and can be found in the appendix of Chiarella et al., 2006). The value of v_t in (CV-1) can quite as well be viewed as arising from a geometric series. Putting $\tau = 1$ (as in Gaunersdorfer, 2000a, p. 806), it is easily checked that the middle row in (CV-1) is equivalent to

$$
\begin{aligned}
m_{\xi t} &= (1-\alpha) \sum_{j=1}^{\infty} \alpha^{j-1} \, (p_{t-j} - R p_{t-j-1})^2 \\
v_t &= (1-\alpha) \sum_{j=1}^{\infty} \alpha^{j-1} \, (p_{t-j} - R p_{t-j-1} - m_{\xi,t-j})^2
\end{aligned}
\tag{12}
$$

Hence the different coefficients $(1-\alpha_v)$ and $\alpha_v \, (1-\alpha_v)$ on the squared deviations in (CV-1) and (CV-2) can be explained by the use of the sample mean m in the geometric series for v_t, which in each term in (12) has the same historic date as the (leading) price related to it, whereas in (11) m is fixed at the date of its most recent computation.

It is interesting to note that the RiskMetrics$^{\mathrm{TM}}$ group in their technical report (Longerstaey et al., 1996, Ch. 5.2), which is a much cited practitioner's guide to measure market risks in portfolios, likewise proposes an exponentially weighted moving average of squared returns as an estimate of volatility. Hence, apart from the fact that their returns are the log price changes whose long-run average is conveniently set equal to zero, they put forward the same formula for v_t as (CV-1). A later version of the report also mentions an optimal decay factor of $\alpha_v \approx 0.94$ for daily data that minimizes the mean squared differences between v_t and the actual squared returns (the value is drawn from Fleming et al., 2001, pp. 333f).

Chartists beliefs employing the concept of a trend estimation as in (CB-2a) can give rise to a treatment of endogenous variations of their conditional variances that is in the same spirit as (CV-2); see Franke and Sethi (1993, 1998). Adjusting the notation to (CV-2) and recalling that their trend is the slope of a straight line fitted through past prices over a rolling sample period of length T_c, the variances of fundamentalists and chartists are determined by

$$\begin{aligned} \sigma^2_{ft} &= \sigma^2_f \;=\; \text{const} \\ \sigma^2_{ct} &= \text{const} \;+\; \alpha_\sigma\, v_t \\ v_t &= \text{variance of the residuals from (CB-2a) in regressing prices on time} \end{aligned} \qquad \text{(CV-2a)}$$

This specification has been categorized as (CV-2a) since the variance of the residuals over the sample period T_c corresponds to the (infinite) sum of the discounted squared price deviations in (11), which we have seen is equivalent to (CV-2).

As already indicated, Franke and Sethi make it sufficiently clear that the variability of σ^2_{ct} is introduced in order to put a curb on the otherwise diverging tendencies in the price dynamics. Chiarella, Dieci and Gardini (2002, p. 176) have the same motivation to postulate a functional relationship between the conditional variance of chartists and past prices. They, too, work with the concept q_t of a price trend, which as formulated in (CB-2) is determined by an adaptive expectations mechanism.[13] It is here supposed that as the price trend becomes steeper, i.e. $|q_t|$ increases, chartists expect greater volatility in prices and so increase their variance. Representing this relationship by a function $v(\cdot)$, we have (for lags $\tau = 0, 1$)

$$\begin{aligned} \sigma^2_{ft} &= \sigma^2_f \;=\; \text{const} \\ \sigma^2_{ct} &= v(|q_t|)\;, \qquad v(\cdot) > 0,\; v'(\cdot) > 0 \\ q_t &= q_{t-1} \;+\; \alpha_q\,(p_{t-\tau} - p_{t-\tau-1} - q_{t-1}) \qquad (0 < \alpha_q < 1) \end{aligned} \qquad \text{(CV-2b)}$$

Given the belief function $B_c = B_c(q_t) = \gamma q_t$, the function $v(\cdot)$ is furthermore supposed to yield a well-defined limit $z_{ct} = E_{ct}(\xi_{t+1})/a\sigma^2_{ct} = \gamma q_t / a\sigma^2_{ct}(|q_t|) = 0$ as q_t tends to zero, and an S-shaped excess demand function $z_{ct} = z_{ct}(q_t)$.

The risk measures so far are based on the observed volatility of prices. An alternative approach evaluates the risk that is directly associated with the two forecasting rules. In essence, it is specified as the discounted infinite sum of the agents' forecast errors in the past,

[13] See the short discussion of (CB-2a) in Section 2.7 for the conceptual relationship between these adaptive expectations and the regressions underlying (CV-2a).

$$
\begin{aligned}
\sigma_{ct}^2 &= v_{ct} \\
\sigma_{ft}^2 &= v_{ft} \,/\, [\,1 + (p_t - p_t^f)^2\,] \\
v_{ht} &= (1-\alpha_v) \sum_{j=1}^{\infty} \alpha_v^j \,[E_{h,t-j}(p_{t+1-j}) - p_{t+1-j}]^2 \,, \qquad h = c, f
\end{aligned} \tag{CV-3}
$$

(setting the lag τ to zero). It will be immediately noted that if the discounting in the infinite sum is done with α_v^{j-1} instead of α_v^j, the variance v_{ht} can also be determined by the usual adaptive expectations mechanism,

$$
v_{ht} \;=\; \alpha_v \, v_{h,t-1} \;+\; (1-\alpha_v)\,[E_{h,t-1}(p_t) - p_t]^2
$$

The dampening term $1//\,[1 + (p_t - p_t^f)^2]$ for σ_{ft}^2 takes account of the different risk that the two groups of traders are facing. In particular, measuring the risk of the fundamentalists by v_{ft} would imply that the more prices deviate from fundamentals, the riskier the fundamentalists would perceive their forecasts to be, which is quite implausible. It seems more reasonable to assume that as the degree of the misalignment grows, the confidence in the fundamentalist expectation rule will improve. Correspondingly, the dampening factor captures the idea that the fundamentalists will then attach increasingly less importance to the (basically short-term) volatility v_{ft}.[14]

An unhandsome feature of (CV-3) is its implication for a deterministic equilibrium point, where both σ_{ft} and σ_{ct} are zero. In their specific models, De Grauwe and Grimaldi use the market equation (MP-1) with zero supply z_{st}, so that only the ratio σ_{ft}/σ_{ct} is relevant. However, an analytical treatment of the limit of this ratio does not seem possible, which also precludes an eigen-value analysis of the system's stability. On the other hand, the authors assert that in their dynamic simulations (with a stable equilibrium) the ratio always converged to unity; see De Grauwe, Dieci and Grimaldi (2005, pp. 24, 40) or De Grauwe and Grimaldi (2006, pp. 10f, 31).

As a concluding remark it may be mentioned that a constant conditional variance (CV-0) of fundamentalists and chartists is often preferred for the obvious reason of simplicity. Chiarella and He (2003, fn 5 on pp. 508, 529) additionally argue that the time-varying variances would not essentially alter the original results under (CV-0). To support this point they refer to Gaunersdorfer (2000a) with a scenario of Walrasian equilibrium prices, and their own work with nonzero excess demand and a market maker. Since these two models also have other features in common (in particular, they both adopt the fundamental value as their base price), one may ask if the essential similarity of the results would still hold when these features are modified; quite apart

[14] It should be added that De Grauwe and Grimaldi (2005, 2006), who have introduced the risk measures in (CV-3), do not use them as conditional variances in the formulation of demand. The only purpose of these σ_{ht}^2 is to specify the risk-adjusted profits in the evolutionary part of their model; see (NP-2) in Section 3.2 below.

from a precision of what is meant by "similar". The role of time-varying conditional variances and the possible effects they produce are thus still an open issue.

3 The evolutionary process

In most models in the BH framework the fractions of fundamentalists and chartists change over time in an endogenous manner. Nevertheless, with suitable nonlinearities the dynamics can already be rich enough when the population shares (PS) remain fixed. So we begin with the basic distinction,

$$n_{ft} \;=\; \bar{n}_f \;=\; \text{const}\;, \qquad n_{ct} \;=\; \bar{n}_c \;=\; \text{const} \tag{PS-0}$$

$$n_{ft}, n_{ct} \;\; \text{vary over time} \tag{PS-1}$$

The common idea in the class of models (PS-1) is that of evolutionary dynamics, where the population shares evolve according to a measure of fitness. Generally, a population share increases if the fitness of the corresponding strategy increases. Over the past decade, however, a great variety of specification details have here developed; model builders actually seem to have spent more creativity on this issue than on the forecasting rules of chartists. In an attempt to tell the details apart, which are partly exclusive and partly supplementary, we now list several (partly hierarchical) topics, where each one contains two or more subcases.

3.1 Gross profits

The fitness measure derives from the realized profits fundamentalists and chartists have earned in the past. We may remark that this is less obvious than it might seem at first sight. Since the agents solve the same optimization problem and only differ in their belief functions, low forecasting errors may be another meaningful fitness criterion. We are led back to this issue below.

Regarding profits in the fitness measure, one finds both gross profits and a concept of net profits in the literature. Beginning with gross profits, a distinction has to be made between normalized and non-normalized profits. That is, the gross profits (GP) of period t for an agent of type h, π^g_{ht}, may be specified as the total profits obtained from his recent purchases or sales $z_{h,t-1}$ of the asset, or the profits per unit invested. Thus, if also a constant cost C_h for adopting the underlying belief function is included,

$$\pi^g_{ht} \;=\; (p_t + y_t - Rp_{t-1})\, z_{h,t-1} \;-\; C_h \tag{GP-1}$$

$$\pi^g_{ht} \;=\; (p_t + y_t - Rp_{t-1})\, \text{sgn}(z_{h,t-1}) \;-\; C_h \tag{GP-2}$$

sgn(·) is the sign function with $\text{sgn}(x) = 1,\ 0,\ -1$ for $x > 0,\ x = 0,\ x < 0$, respectively. Profits are positive (negative) if agents have bought (sold) the asset and excess returns turned out to be positive (negative) as expected, while they are negative if expected and realized excess returns have opposite signs. The idea of C_h is that fundamentalism might be more profitable in the long-run, but on the other hand it is also more costly to gather information about the fundamental value than to compute the expected price from a simple technical rule. Hence, if the cost terms are introduced into a model, $C_f > C_c \geq 0$ is assumed.

Two reasons are given in justifying (GP-2). First, it is pointed out that this criterion abstracts from the agents' wealth. For example (De Grauwe and Grimaldi, 2004, p. 8; 2006, p. 8), "suppose that technical traders happen to have more wealth than fundamentalists so that their total profits exceed the fundamentalists' profits despite the fact that the technical rule happens to be less profitable (per unit invested) than the fundamentalist rule. In this case, our switching rule will select the fundamentalist rule although the agents who use this rule make less profits (because their wealth happens to be small) than agents using chartist rules." This reasoning is not pertinent in our opinion, since none of the models adopting the BH framework (with CARA utility function, in technical terms) keeps track of the actual or potential wealth of an agent or agent type, including the many models using (GP-1). In addition, should a strategy that independently of an agent's level of wealth (as a consequence of the CARA assumption) invests 100,000\$ in an asset yielding a return of 1% not have more adherents than a strategy earning 10%, which because of risk-aversion or its price expectations only invests 1,000\$?

According to the second argument in favour of (GP-2), an agent considering to switch from fundamentalism to chartism, say, only needs to to use publicly available information, i.e., the observed price change and the forecasting rule from which $z_{c,t-1}$ has derived. This equally holds true for (GP-1), but it may be a point of concern further below.

3.2 Net profits

Often the profits from (GP-1) or (GP-2) are "discounted" for the risk associated with them. This kind of profits may be called net profits (NP), denoted as π^n_{ht} for agents of type h. Models that do not take this option can be referred to as (NP-0),

$$\pi^n_{ht} \quad = \quad \pi^g_{ht} \tag{NP-0}$$

It may be argued that in real markets realized profits or accumulated wealth is what investors care about most, which suggests that (NP-0) would be practically more important than any risk-adjusted measure of profits (Hommes, 2001, p. 156). On the other hand, evaluating the fitness of fundamentalism or chartism on the basis of (NP-0) is inconsistent with the feature that all agents are mean-variance maximizers (Gaunersdorfer, 2000b, p. 13). Nevertheless, (NP-0) need not necessarily be discarded on that account. It just means that the mean-variance utility should not be taken literally, and that it only serves to obtain a rough-and-ready determination of the agents' demand. In this case, however, one may quite as well take one step further: abandon the maximization problem (2) altogether and start out directly from eq. (3), $z_{ht} = E_{ht}(\xi_{t+1})/a\sigma^2_{ht}$, as a reasonable formulation of demand.

If agents are conceived as mean-variance optimizers proper, net profits are identified with the utility function (2). Correspondingly, gross profits are given by (GP-1) and we have (for $h = c, f$)

$$\pi^n_{ht} \;=\; \pi^g_{ht} \;-\; \frac{a\,\sigma^2_{h,t-1}}{2}\, z^2_{h,t-1} \tag{NP-1}$$

In the models where the conditional variances are supposed to be uniform, $\sigma^2_{f,t-1} = \sigma^2_{f,t-1} = \sigma^2_{t-1}$, (NP-1) together with (GP-1) has an interesting relationship to the forecasting errors. Under these circumstances fundamentalism and chartism only differ in the way in which they form their price expectations. Hence small forecast errors are a straightforward criterion to evaluate the performance of the two types of behaviour. The intuition, of course, is that agents with better predictions also make more money; after discounting for the risk in the optimization calculus, it should be added. The following argument provides a rigorous basis for this idea and ends up with an alternative formulation of (NP-1).[15]

To begin with, note that by virtue of $\pi^n_{ht} = \pi^g_{ht} - a\sigma^2_{t-1}\, z^2_{h,t-1}/2 = \xi_t\, z_{h,t-1} - a\sigma^2_{t-1}\; z^2_{h,t-1}/2 - C_h$, the maximization problem (2) of an agent of type h dated one period backward can be written as a maximization of $E_{h,t-1}[\pi^n_{ht}] = E_{h,t-1}[\pi^n_{ht}(z_h)]$. The solution is not affected by the predetermined cost C_h, and we may quite as well subtract another constant in the objective function. Let this term be the risk-adjusted profit $\pi^n_{pf,t}$ of a hypothetical agent with perfect foresight and the same conditional variance σ^2_{t-1} (but without a cost term). It is given by

$$\pi^n_{pf,t} \;=\; \xi_t\,\frac{\xi_t}{a\sigma^2_{t-1}} \;-\; \frac{a\sigma^2_{t-1}}{2}\,\frac{\xi^2_t}{a^2\,\sigma^4_{t-1}} \;=\; \frac{\xi^2_t}{2a\sigma^2_{t-1}}$$

As it should be, a perfectly rational agent would always make positive profits if excess returns are non-zero, even after discounting for the risk. These profits are reduced as the risk perception σ^2_{t-1} increases.

To proceed with the argument, an agent of type h could now be treated as if he maximizes the expected value of the objective function $\tilde{\pi}^n_{ht} = \tilde{\pi}^n_{ht}(z_h) := \pi^n_{ht}(z_h) - \pi^n_{pf,t}$. A simple calculation

[15] The argument is borrowed from Hommes (2001, p. 156).

shows that the (old and new) solution $z_{h,t-1} = E_{h,t-1}(\xi_t)/(a\sigma^2_{t-1})$ yields a realized 'utility'

$$\begin{aligned} \tilde{\pi}^n_{ht} \;=\; \pi^n_{ht} - \pi^n_{pf,t} \;&=\; \frac{-1}{2a\,\sigma^2_{t-1}} \Big\{ \xi_t \,-\, E_{h,t-1}(\xi_t) \Big\}^2 \,-\, C_h \\ &=\; \frac{-1}{2a\,\sigma^2_{t-1}} \Big\{ [p_t - E_{h,t-1}(p_t)] \,+\, [y_t - E_{h,t-1}(y_t)] \Big\}^2 \,-\, C_h \end{aligned} \tag{13}$$

The expression $\tilde{\pi}^n_{ht}$ is, of course, nonpositive, since the net profits of agent h are compared to the net profits that could have been maximally earned. It might thus also be said that $z_{h,t-1}$ minimizes the disutility $-\tilde{\pi}^n_{ht}(z_h)$.

To sum up, given that gross profits are not normalized and thus specified by (GP-1), an agent of type h who fares well with π^n_{ht} as his utility function, fares equally well with the objective function given by $\tilde{\pi}^n_{ht}$. Dropping the tilde and recalling the correct expectations of dividends by all agents, $E_{h,t-1}(y_t) = E_{t-1}(y_t) = \bar{y}$, the criterion (NP-1) can be equivalently replaced with

$$\pi^n_{ht} \;=\; \frac{-1}{2a\,\sigma^2_{t-1}} \Big\{ [p_t - E_{h,t-1}(p_t)] \,+\, [y_t - \bar{y}] \Big\}^2 \,-\, C_h \tag{NP-1a}$$

(NP-1) in conjunction with (GP-1) and the assumption $\sigma^2_{f,t-1} = \sigma^2_{f,t-1} = \sigma^2_{t-1}$ will be referred to as (NP-1a) if the authors make the equivalence just discussed explicit.

Taken on its own, the concept could also admit differentiated conditional variances $\sigma^2_{h,t-1}$, if the profits in $\pi^n_{pf,t}$ are discounted with the same variance. Then, however, one would argue with two perfectly rational agents with two different risk perceptions. More importantly, (NP-1) and (NP-1a) would no longer yield the same results in the model's discrete choice component below; see the remark on eq. (14) there presented.

Another notion of risk-adjusted profits applies if the gross profits π^g_{ht} are specified as unit profits. In this case a normalized measure of risk has to be subtracted from the unit returns in (GP-2). In DeGrauwe and Grimaldi (2006, p. 8) this is the conditional variance itself, multiplied by the risk aversion coefficient a, so that this version of net profits π^n_{ht} (for $h = c, f$) reads,

$$\pi^n_{ht} \;=\; \pi^g_{ht} \,-\, a\,\sigma^2_{ht} \tag{NP-2}$$

Although π^g_{ht} and $a\sigma^2_{ht}$ are at conceptually compatible levels, the simulations of a numerically specified model should check that the two terms are of a similar order of magnitude. Otherwise it would be more transparent to identify the relevant fitness measure directly with the dominating term in (NP-2).

3.3 Memory in the fitness measure

Measuring the fitness of a belief function B_h by realized profits, with or without risk adjustment, could induce an unduly high volatility in the agents' population shares, if the prices do not behave too regularly. As this may (and should) easily happen in a model with daily prices, it seems then more reasonable that agents consider the performance of a belief function over a longer span of time, where older profits may be discounted in some way. This idea usually runs under the heading of *memory* in the evolutionary fitness (MF). Denoting the period-t fitness assigned to an agent of type h by U_{ht}, we have the following distinction:

$$U_{ht} = \pi^n_{ht} \tag{MF-0}$$

$$U_{ht} = \pi^n_{ht} + \eta\, U_{h,t-1} \qquad (0 < \eta \leq 1) \tag{MF-1}$$

Memory is measured by the coefficient η, which indicates how fast past realized fitness is discounted for the selection of a belief function (this is spelled out in a moment). Memory is shortest for the extreme case $\eta = 0$, which would yield (MF-0). In the other polar case, $\eta = 1$, the fitness is given by the accumulated net profits over the entire past. Under (GP-1) where the profits are not risk-adjusted, this corresponds to the total wealth of a hypothetical agent who would have persistently stuck to one type of expectation formation.

Regarding the informational issues behind (MF-1) it may be noted that a fundamentalist considering to convert to chartism need not necessarily know a chartist's fitness $U_{c,t-1}$ from the previous period. Dividing $\pi^n_{ht} = \pi^n_{ct}$ by $1-\eta$ (given that $\eta < 1$), the equation (MF-1) can be rewritten as $U_{ht} = (1-\eta)\tilde{\pi}^n_{ht} + \eta U_{h,t-1}$. Since this is an adaptive expectations mechanism, such an agent could also use the equivalent formula $U_{ht} = (1-\eta) \sum_{k=0}^{\infty} \eta^k \, \tilde{\pi}^n_{h,t-\tau}$, which only requires him to know the record of past prices and accept a negligible approximation error if the record is finite (time-varying conditional variances σ_{ht}, likewise, depend on past prices only; see below).

3.4 Discrete choice

The fitness measure in the preceding section provides the basis for updating the population shares. According to the concept of evolutionary fitness, the higher the fitness associated with a belief function, the more agents should select this function. To this end the discrete choice (DC) model is employed, where the population shares derive from the multinomial logit (or 'Gibbs') probabilities.[16] In a couple of models in the BH framework these fractions are subsequently still somewhat

[16] The standard reference for an extensive discussion of this approach are Manski and McFadden (1981), and Anderson et al. (1993).

modified, so we denote them as $n^\star_{ht}$. Two cases may nevertheless be already here distinguished, where $n^\star_{ht}$ is to be thought of as being determined at the beginning of period t:

$$n^\star_{ct} \;=\; \hat{n}_{ct} \;:=\; \frac{\exp(\beta\, U_{c,t-1})}{\exp(\beta\, U_{f,t-1}) + \exp(\beta\, U_{c,t-1})}\;, \qquad n^\star_{ft} = 1 - n^\star_{ct} \qquad \text{(DC-0)}$$

$$n^\star_{ct} \;=\; \hat{n}_{ct}\; \exp[\,-(p_{t-1} - p^f_{t-1})^2/\theta\,]\;, \qquad n^\star_{ft} = 1 - n^\star_{ct} \qquad (\theta > 0) \qquad \text{(DC-1)}$$

(assuming a lag $\tau = 1$ in the second equation). In almost all of the models the parameter β is of crucial importance for their dynamic properties. It is called the *intensity of choice* and measures the agents' responsiveness to the fitness differentials when choosing their belief function. In the extreme case $\beta = 0$, differences in the fitness cannot be observed or are ignored, so that equal fractions $n^\star_{ct} = n^\star_{ft} = 1/2$ are obtained. An increase in β can be said to represent rising evolutionary pressure, or an increase in the degree of rationality with respect to evolutionary selection of trading strategies. With an infinite responsiveness, $\beta = \infty$, all agents select the most successful belief function and are thus all either fundamentalists or chartists (if $U_{c,t-1} \neq U_{f,t-1}$).

Unfortunately, intermediate values of β are generally hard to compare across different models. In order to assess whether the evolutionary pressure is stronger in one model than in another, one needs to consider the scale of the fitness measure. This, in turn, varies with the scale of prices in the gross profit expression, which is determined by the numerical specification of the (initial or fixed) fundamental price. So far, two intensities of choice could still be made comparable by multiplying one of them with a suitable factor if the the two models have different fundamental prices (in their deterministic long-run equilibria). This basis for a rescaling, however, is eroded if there is memory in the fitness measure and the $U_{h,t-1}$ represent accumulated profits. Even if they are stationary in the long-run, their average levels and 'amplitudes' will be certainly affected by the memory parameters η employed in the two models.

The conditioning of the chartist fraction upon fundamentals in (DC-1) can be seen as a stabilizing transversality condition in a world with heterogeneous agents (Hommes, 2001, p. 161). Its role is very similar to the transversality condition $\lim_{k\to\infty} E_t(p_{t+k})/R^k = 0$ in a rational expectations world, which is there needed to rule out explosive bubble solutions. The multiplicative term $\exp[\ldots] \leq 1$ in (DC-1) was found useful in models where chartists have a destabilizing influence and occasionally may become dominant in the market, such that prices are growing exponentially. (DC-1) allows the (as it turns out) stabilizing fundamentalists to gain a greater weight then and so put a curb on the diverging tendencies. Conceptually, the "penalty" in the evaluation of the fitness could be interpreted as representing a decreasing confidence in the way chartists form their expectations, in a stage of the dynamic process when prices deviate more and more from their benchmark value to which even the most technical traders feel they should eventually return.[17]

In an older paper, Gaunersdorfer (2000a, p. 806) has introduced the penalty directly into the fitness measure, there as a positive term in the fitness of fundamentalists. Because of its closeness to (DC-1), we may report this specification here as

$$U_{ft} \;=\; \pi^n_{ft} \;+\; \eta\, U_{f,t-1} \;+\; \theta\,(p_{t-1} - p^f_{t-1})^2 \qquad \text{in (DC-0)} \qquad (\theta > 0) \qquad \text{(DC-1a)}$$

(again with respect to a one-period lag, $\tau = 1$). Gaunersdorfer actually sets $\eta = 0$. She acknowledges the need for a less ad hoc mechanism that would drive prices back to normal if they have diverged too far, but justifies (DC-1a) by pointing out it would be useful to get more insight into the dynamics generated by a simple device before one adds another layer of complexity (Gaunersdorfer, 2000a, pp. 806f).

What, of course, counts in the discrete choice setting is differential fitness. This is immediately verified by rearranging the terms in (DC-0) as

$$\hat{n}_{ct} \;=\; \frac{1}{1 \;+\; \exp[\,\beta\,(U_{f,t-1} - U_{c,t-1})\,]} \tag{14}$$

The equation makes it, in particular, clear that, with uniform conditional variances σ^2_{t-1}, it does not matter whether $\pi^n_{h,t-1}$ from (NP-1) enters $U_{h,t-1}$ ($h = c, f$) or the hypothetical payoffs $\tilde{\pi}^n_{h,t-1} = \pi^n_{h,t-1} - \pi^n_{pf,t-1}$ from (13) and (NP-1a), which compare agent h's realized net profits to the net profits earned by a hypothetical perfect foresight agent. Obviously, with no memory in the fitness measure as in (MF-0), one gets $U_{f,t-1} - U_{c,t-1} = \tilde{\pi}^n_{f,t-1} - \tilde{\pi}^n_{c,t-1} = \pi^n_{f,t-1} - \pi^n_{pf,t-1} - (\pi^n_{c,t-1} - \pi^n_{pf,t-1}) = \pi^n_{f,t-1} - \pi^n_{c,t-1}$. By backward recursion this mechanism carries over if (MF-1) applies with positive memory parameter $\eta > 0$. Observe that the argument does indeed rely on the homogeneity assumption $\sigma^2_{f,t-1} = \sigma^2_{c,t-1} = \sigma^2_{t-1}$. Otherwise, as mentioned in the comment on (NP-1a), we would have two different expressions for $\pi^n_{pf,t-1}$ in the differential $\tilde{\pi}^n_{f,t-1} - \tilde{\pi}^n_{c,t-1}$, one for fundamentalists and one for chartists, and the two would no longer cancel out.

3.5 Inertia in the evolutionary process

In the first generations of models in the BH framework, the population shares were directly determined by the discrete choice model, without any further modifications or inertia (IN). In our

[17] The basic idea is usually ascribed to the modelling of the Santa Fe artificial stock market. There the agents' predictors are more general than our belief functions in that they are condition-forecast rules of the form: if conditions A, B, C, ... about past prices are satisfied, then espect tomorrow's price to be determined by rule X; see Arthur et al. (1997, pp. 24f) or LeBaron et al. (1999, pp. 1493ff).

notation, the market fractions n_{ct} and n_{ft} of chartists and fundamentalists over period t (between t and $t{+}1$) are then given by

$$n_{ct} \;=\; n^{\star}_{ct} \;, \qquad n_{ft} \;=\; n^{\star}_{ft} \tag{IN-0}$$

If the population shares react to the most recent differential profits (i.e., there is no or only low memory in the fitness measure) and the stabilizing forces are relatively weak in certain stages of the dynamic process, then prices might diverge unrealistically fast and too far from the fundamental value. In any case, one may question the credibility of a model in which either fundamentalism or chartism has completely disappeared from the market, and these states even alternate quite rapidly. This concern has probably been the reason for making one of the following amends.

A straightforward way to avoid wide fluctuations of the population shares is to impose an upper-bound on them, which introduces a first and very simple type of inertia into the evolutionary mechanism. Accordingly, let $\bar{n}_{ct}$ and $\bar{n}_{ft}$ be the fixed proportions of agents who are and remain, respectively, chartists and fundamentalists all of their market life. $1 - \bar{n}_{ct} - \bar{n}_{ft}$ is the fraction of agents, called adaptively rational agents, who may switch. In this setting, $n^{\star}_{ht}$ determined in (DC-0) or (DC-1) is the proportion of agents of type h among these adaptively rational agents. The total number of agents of type h, expressed as a percentage of the total population, is thus determined as

$$n_{ht} \;=\; \bar{n}_{ht} \;+\; (1 - \bar{n}_{ct} - \bar{n}_{ft})\, n^{\star}_{ht} \;, \qquad h = c, f \tag{IN-1}$$

The condition $n_{ct}{+}n_{ft} = 1$ is easily checked, and $1{-}\bar{n}_{ct}$ $(1{-}\bar{n}_{ft})$ is the maximal share of fundamentalism (chartism) in the market. Obviously, this device limits the stabilizing (destabilizing) forces in systems where increasing shares of fundamentalism (chartism) turn out to reinforce the converging (diverging) tendencies. One may hope that this prevents prices from moving too rapidly away from and back toward fundamentals.

Another idea of inertia is asynchronous updating, which may be introduced for similar reasons or just for greater realism in describing the agents' adaptive behaviour. Here it is supposed that in each period only a part of the agents evaluate the two fitness measures and consider to switch. This is readily captured by the partial adjustment equation,

$$n_{ht} \;=\; \nu\, n_{h,t-1} \;+\; (1 - \nu)\, n^{\star}_{ht} \;, \qquad h = c, f \qquad\qquad (0 < \nu < 1) \tag{IN-2}$$

The coefficient ν measures the stickiness in the updating mechanism; with $\nu{=}0$ we would be back in (IN-0), and $\nu{=}1$ represents maximal stickiness where, as already put forward in (PS-0) above, the agents do not switch at all.

4 Application of the classification scheme

4.1 Succinct review of the specification details

The previous Sections 2 and 3 have discussed at some length various aspects of asset pricing models in the tradition of Beja–Goldman and Brock–Hommes. In this section we choose a larger number of papers from the literature and classify them according to the scheme we have set up. For a better overview, we first collect all the distinctions we made on the next few pages.

Total supply:

$$
\begin{aligned}
z_{st} &= 0 \qquad \text{for all } t && \text{(TS-0)}\\
z_{st} &= \bar{z}_s \neq 0 \qquad \text{for all } t && \text{(TS-1)}\\
z_{st} & \text{ determined by central bank policy} && \text{(TS-2)}
\end{aligned}
$$

Base price:

$$
\begin{aligned}
E_{ht}(p_{t+1}) &= p_{t-\tau} + B_h(p_t^f, p_{[t-\tau,\cdots]}, \cdot\,)\,, \qquad \text{i.e. } p_t^b = p_{t-\tau} && \text{(BP-1)}\\
E_{ht}(p_{t+1}) &= p_t^f + B_h(p_t^f, p_{[t-\tau,\cdots]}, \cdot\,)\,, \qquad \text{i.e. } p_t^b = p_t^f && \text{(BP-2)}
\end{aligned}
$$

Market price:

$$
\begin{aligned}
p_t &= \frac{1}{R}\left[p_{t-1} + \bar{y} + \frac{n_{ft}\, B_f(p_t^f, p_{[t-1,\cdots]})}{\tilde{n}_t\, \sigma_{ft}^2} + \frac{n_{ct}\, B_c(p_t^f, p_{[t-1,\cdots]})}{\tilde{n}_t\, \sigma_{ct}^2} - \frac{a\, z_{st}}{\tilde{n}_t}\right]\\
\tilde{n}_t &:= \frac{n_{ft}}{\sigma_{ft}^2} + \frac{n_{ct}}{\sigma_{ct}^2} && \text{(MP-1)}
\end{aligned}
$$

$$
p_t = \frac{1}{R}\left[p_{t-1} + \bar{y} + n_{ft}\, B_f(p_t^f, p_{[t-1,\cdots]}) + n_{ct}\, B_c(p_t^f, p_{[t-1,\cdots]})\right] \qquad \text{(MP-1a)}
$$

$$
p_t = p_t^f + \frac{1}{R}\left[n_{ft}\, B_f(p_t^f, p_{[t-1,\cdots]}) + n_{ct}\, B_c(p_t^f, p_{[t-1,\cdots]})\right] \qquad \text{(MP-1b)}
$$

$$
\begin{aligned}
p_{t+1} &= p_t + \mu\left\{\sum_{h=f,c} \frac{n_{ht}\,[p_t + \bar{y} + B_h(p_t^f, p_{[t,\cdots]}) - Rp_t]}{a\, \sigma_{ht}^2}\right\} && \text{(MP-2)}\\
p_{t+1} &= p_t + \mu\left\{\sum_{h=f,c} \frac{n_{ht}\,[p_t^f + \bar{y} + B_h(p_t^f, p_{[t,\cdots]}) - Rp_t]}{a\, \sigma_{ht}^2}\right\} && \text{(MP-3)}\\
\mu &= \mu_t = \frac{\bar{\mu}}{\bar{\mu}_o + (n_{ct}|z_{ct}| + n_{ft}|z_{ft}|)^{\bar{\mu}_1}} \qquad \text{in (MP-2,3)} && \text{(MP-4)}
\end{aligned}
$$

Fundamental value:

$$
\begin{aligned}
p_t^f &= p^f = \text{const} \qquad \text{for all } t && \text{(FV-0)}\\
p_t^f &= p_{t-1}^f + \varepsilon_t\,, \qquad \varepsilon_t \sim N(0, \sigma_{pf}^2) && \text{(FV-1)}\\
p_t^f &= (1+\varepsilon_t)\, p_{t-1}^f\,, \qquad \varepsilon_t \sim N(0, \sigma_{rf}^2) && \text{(FV-1)}\\
p_t^f &= p_{t-1}^f + \varepsilon_t\,, \qquad \varepsilon_t = \varepsilon_t^o \sqrt{\alpha_{\varepsilon 0} + \alpha_{\varepsilon 1}\, \varepsilon_{t-1}^2}\,, \qquad \varepsilon_t^o \sim N(0,1) && \text{(FV-2)}
\end{aligned}
$$

Fundamentalist beliefs:

$$B_f = B_f(p_t^f, p_{t-\tau}) = \phi\,(p_t^f - p_{t-\tau}) \qquad (0 \le \phi \le 1) \tag{FB-1}$$

$$B_f = B_f(p_t^f, p_{t-\tau}) = \begin{cases} \phi\,(p_t^f - p_{t-\tau}) & \text{if } |p_{t-\tau} - p_t^f| > C \\ 0 & \text{else} \end{cases} \tag{FB-2}$$

Chartist beliefs:

$$B_c = B_c(p_{[t-\tau,\cdots]}) = \gamma \sum_{j=0}^{T_c-1} \alpha_{\gamma j}\,(p_{t-\tau-j} - p_{t-\tau-j-1}) \qquad (\textstyle\sum_j \alpha_{\gamma j} = 1) \tag{CB-1}$$

$$B_c = B_c(p_{t-\tau}, p_{t-\tau-1}) = \gamma\,(p_{t-\tau} - p_{t-\tau-1}) \tag{CB-1a}$$

$$\begin{aligned} q_t &= q_{t-1} + \alpha_q\,(p_{t-\tau} - p_{t-\tau-1} - q_{t-1}) \qquad (0 < \alpha_q < 1) \\ B_c &= B_c(q_t) = \gamma\, q_t \end{aligned} \tag{CB-2}$$

$$\begin{aligned} q_t &= \text{slope of simple regression of } p_{t-\tau}, p_{t-\tau-1}, \ldots, p_{t-T_c} \text{ against time} \\ B_c &= B_c(q_t) = \gamma\, q_t \end{aligned} \tag{CB-2a}$$

$$\begin{aligned} m_t &= \sum_{j=0}^{T_c-1} \alpha_{\gamma j}\, p_{t-\tau-j} \\ B_c &= B_c(p_t^f, p_{[t-\tau,\cdots]}) = \gamma\,(m_t - p_t^f) \end{aligned} \tag{CB-3}$$

$$B_c = B_c(p_t^f, p_{t-\tau}) = \gamma\,(p_{t-\tau} - p_t^f) \tag{CB-3a}$$

$$\begin{aligned} m_t &= \sum_{j=0}^{T_c-1} \alpha_{mj}\, p_{t-\tau-j} \qquad (\textstyle\sum_j \alpha_{mj} = 1) \\ B_c &= B_c(p_{t-\tau}, m_t) = \gamma\,(p_{t-\tau} - m_t) \end{aligned} \tag{CB-4}$$

$$\begin{aligned} m_t &= \alpha_m\, m_{t-1} + (1-\alpha_m)\, p_t \qquad (0 \le \alpha_m \le 1) \\ B_c &= B_c(p_t, m_t) = \gamma\,(p_t - m_t) \end{aligned} \tag{CB-4a}$$

Conditional variance:

$$\sigma^2_{ft} \;=\; \sigma^2_{ct} \;=\; \sigma^2 \;=\; \text{const} \tag{CV-0}$$

$$\begin{aligned}
\sigma^2_{ht} &= \text{Var}_{ht}(p_{t+1} - Rp_t) \;+\; \sigma^2_y \;=\; v_t + \sigma^2_y \,, \qquad h = c, f \qquad (\sigma^2_y = \text{const}) \\
v_t &= \alpha_v\, v_{t-1} \;+\; (1-\alpha_v)\,(p_{t-\tau} - Rp_{t-\tau-1} - m_{\xi,t-1})^2 \qquad \text{(CV-1)} \\
m_{\xi t} &= \alpha_m\, m_{\xi,t-1} \;+\; (1-\alpha_m)\,(p_{t-\tau} - Rp_{t-\tau-1}) \qquad (0 \le \alpha_v, \alpha_m \le 1)
\end{aligned}$$

$$\begin{aligned}
\sigma^2_{ft} &= \text{Var}_{ft}(p_{t+1}) \;+\; \sigma^2_y \;=\; \sigma^2_p + \sigma^2_y \qquad (\sigma^2_p, \sigma^2_y = \text{const}) \\
\sigma^2_{ct} &= \text{Var}_{ct}(p_{t+1}) \;+\; \sigma^2_y \;=\; \sigma^2_p + \alpha_\sigma v_t + \sigma^2_y \\
v_t &= \alpha_v\, v_{t-1} \;+\; \alpha_v\,(1-\alpha_v)\,(p_{t-\tau} - m_{t-1})^2 \qquad \text{(CV-2)} \\
m_t &= \alpha_m\, m_{t-1} \;+\; (1-\alpha_m)\, p_{t-\tau} \qquad (0 \le \alpha_v = \alpha_m < 1)
\end{aligned}$$

$$\begin{aligned}
\sigma^2_{ft} &= \sigma^2_f \;=\; \text{const} \\
\sigma^2_{ct} &= \text{const} \;+\; \alpha_\sigma\, v_t \qquad \text{(CV-2a)} \\
v_t &= \text{variance of the residuals from (CB-2a) in regressing prices on time}
\end{aligned}$$

$$\begin{aligned}
\sigma^2_{ft} &= \sigma^2_f \;=\; \text{const} \\
\sigma^2_{ct} &= v(|q_t|) \,, \qquad v(\cdot) > 0,\; v'(\cdot) > 0 \qquad \text{(CV-2b)} \\
q_t &= q_{t-1} \;+\; \alpha_q\,(p_{t-\tau} - p_{t-\tau-1} - q_{t-1}) \qquad (0 < \alpha_q < 1)
\end{aligned}$$

$$\begin{aligned}
\sigma^2_{ct} &= v_{ct} \\
\sigma^2_{ft} &= v_{ft} \,/\, [\,1 + (p_t - p^f_t)^2\,] \qquad \text{(CV-3)} \\
v_{ht} &= (1-\alpha_v) \sum_{j=1}^{\infty} \alpha_v^j\, [E_{h,t-j}(p_{t+1-j}) - p_{t+1-j}]^2 \;,
\end{aligned}$$

Population shares:

$$n_{ft} \;=\; \bar{n}_f \;=\; \text{const}\;, \qquad n_{ct} \;=\; \bar{n}_c \;=\; \text{const} \tag{PS-0}$$

$$n_{ft}, n_{ct} \;\; \text{vary over time} \tag{PS-1}$$

Gross profits:

$$\pi^g_{ht} \;=\; (p_t + y_t - R p_{t-1})\, z_{h,t-1} \;-\; C_h \tag{GP-1}$$

$$\pi^g_{ht} \;=\; (p_t + y_t - R p_{t-1})\, \text{sgn}(z_{h,t-1}) \;-\; C_h \tag{GP-2}$$

Net profits:

$$\pi^n_{ht} \;=\; \pi^g_{ht} \tag{NP-0}$$

$$\pi^n_{ht} \;=\; \pi^g_{ht} \;-\; \frac{a\,\sigma^2_{h,t-1}}{2}\, z^2_{h,t-1} \tag{NP-1}$$

$$\pi^n_{ht} \;=\; \frac{-1}{2a\,\sigma^2_{t-1}} \Big\{ [p_t - E_{h,t-1}(p_t)] \;+\; [y_t - \bar{y}] \Big\}^2 \;-\; C_h \tag{NP-1a}$$

$$\pi^n_{ht} \;=\; \pi^g_{ht} \;-\; a\,\sigma^2_{ht} \tag{NP-2}$$

Memory in the evolutionary fitness:

$$U_{ht} \;=\; \pi^n_{ht} \tag{MF-0}$$

$$U_{ht} \;=\; \pi^n_{ht} \;+\; \eta\, U_{h,t-1} \qquad\qquad (0 < \eta \leq 1) \tag{MF-1}$$

Discrete choice:

$$n^\star_{ct} \;=\; \hat{n}_{ct} \;:=\; \frac{\exp(\beta\, U_{c,t-1})}{\exp(\beta\, U_{f,t-1}) + \exp(\beta\, U_{c,t-1})}\;, \qquad\qquad n^\star_{ft} = 1 - n^\star_{ct} \tag{DC-0}$$

$$n^\star_{ct} \;=\; \hat{n}_{ct}\;\exp[\,-(p_{t-1} - p^f_{t-1})^2/\theta\,]\;, \qquad\qquad n^\star_{ft} = 1 - n^\star_{ct} \qquad (\theta > 0) \tag{DC-1}$$

$$U_{ft} \;=\; \pi^n_{ft} \;+\; \eta\, U_{f,t-1} \;+\; \theta\,(p_{t-1} - p^f_{t-1})^2 \qquad \text{in (DC-0)} \qquad\qquad (\theta > 0) \tag{DC-1a}$$

Inertia in the evolutionary process:

$$n_{ct} \;=\; n^\star_{ct}\;, \qquad n_{ft} \;=\; n^\star_{ft} \tag{IN-0}$$

$$n_{ht} \;=\; \bar{n}_{ht} \;+\; (1 - \bar{n}_{ct} - \bar{n}_{ft})\, n^\star_{ht}\;, \qquad h = c, f \qquad \bar{n}_{ct}, \bar{n}_{ft} = \text{const} \tag{IN-1}$$

$$n_{ht} \;=\; \nu\, n_{h,t-1} \;+\; (1-\nu)\, n^\star_{ht}\;, \qquad h = c, f \qquad\qquad (0 < \nu < 1) \tag{IN-2}$$

4.2 Assigning the literature

Much of the literature that models speculative asset price dynamics along the lines of Beja–Goldman and Brock–Hommes is conncected to (at least) one of three research centers, in the sense that at least one author of the mostly joint papers is there affiliated. These are departments at the universities in the cities of Amsterdam, Leuven, and Sydney. The authors of the papers we will consider can be assigned to them as follows.

University of Amsterdam: Cars Hommes; A. Gaunersdorfer, H. Huang, F.O.O. Wagner, D. Wang.

University of Leuven: Paul De Grauwe; M. Beine, R. Dieci, M. Grimaldi, R.C.J. Zwinkels.

University of Technology, Sydney: Carl Chiarella; R. Dieci, I. Foroni, L. Gardini, X.-Z. (Tony) He, C. Hommes, Y. Li, D. Wang.

The first authors may be regarded as the 'representative agent' of the corresponding research center. Even under the few authors here mentioned there are obviously some cross relationships between the centers.

We choose seven papers from the Amsterdam research center, five from Leuven and seven from Sydney, numbering them as A1 – A7, L1 – L5, and S1 – S7, respectively. For each paper the modules total supply (TS), base price (BP), etc., are considered and the corresponding variants from the classification scheme are identified. Altogether, these assignments are recorded in Table 1.[18] The references themselves are listed subsequently.

As the table is organized, certain patterns can be easily recognized. The following points may serve as a (very) rough-and-ready characterization of the three "schools".

1. Amsterdam attaches importance to a more elaborated evolutionary process.
2. In contrast, Amsterdam is not very interested in variations of the conditional variance.
3. The most elaborate version of chartist beliefs is used by Sydney.
4. Sydney differs from Amsterdam and Leuven in that its net profits are not risk adjusted.
5. Amsterdam and Leuven differ in the precise specification of their risk-adjusted net profits.

Table 1 and this short characterization of the main emphasis in the work of the three research centers is the upshot of our application of the classification scheme to a wider field of the literature on small-scale asset pricing models.

[18]The question mark in the L3 row indicates that this paper does not make its concept of gross profits explicit. In the context of the other papers there will, however, be no doubt that variant (GP-2) is meant.

	TS	BP	MP	FV	FB	CB	CV	PS	GP	NP	MF	DC	IN
Amsterdam:													
A1	0	1	1a	0	1	1a	0	1	1	1a	1	1	0
A2	0	1	1a	0	1	1a	0	1	1	1a	0	1	0
A3	0	1	1a	0	1	1a	0	1	1	0	1	1	0
A4	0	1	1a	0	1	1a	0	1	1	1a	1	1	0
A5	0	1	1a	0	1	1a	0	1	1	1a	0	1	0
A6	0	2	1b	0	1	3a	1	1	1	1a	0	0	0
A7	1	2	3	0	1	3a	0	1	1	0	1	0	2
Leuven:													
L1	0	1	1	1	1	1	3	1	2	2	0	0	0
L2	2	1	1	1	1	1	3	1	2	2	0	0	0
L3	0	1	2	1	2	1	3	1	?	2	0	0	0
L4	0	1	1	0	2	1	3	1	2	2	0	0	0
L5	1	1	1	2	1	1	3	1	2	2	0	0	0
Sydney:													
S1	0	1	2	1	1	4a	2	0	—	—	—	—	—
S2	0	1	2	1	1	4a	2	0	—	—	—	—	—
S3	0	1	2	0	1	2	2b	0	—	—	—	—	—
S4	0	1	2	1	1	4a	2	1	1	0	0	0	1
S5	0	1	1	1	1	4	2	1	1	0	0	0	0
S6	0	2	3	0	1	3	0	1	1	0	0	0	0
S7	0	1	1a	1	1	4	0	1	1	0	1	0	0

Table 1: Classification of the selected references.

A1: Gaunersdorfer, A., Hommes, C.H. and Wagner, F.O.O. (2006), "Bifurcation routes to volatility clustering under evolutionary learning", *Journal of Economic Behavior and Organization* (forthcoming).

A2: Gaunersdorfer, A., Hommes, C.H. and Wagener, F.O.O. (2003), "Nonlocal onset of instability in an asset pricing model with heterogeneous agents", CeNDEF Working Paper 03–10, University of Amsterdam.

A3: Gaunersdorfer, A. and Hommes, C.H. (2007), "A nonlinear structural model for volatility clustering", In: G. Teyssière and A.P. Kirman (eds.), *Long Memory in Economics*. Berlin: Springer; pp. 265–288. (quoted from CeNDEF Working Paper 05–02, University of Amsterdam).

A4: Gaunersdorfer, A. (2000), "Adaptive beliefs and the volatility of asset prices", *Central European Journal of Operations Research*, 29, 5–30.

A5: Hommes, C.H. (2001), "Financial markets as nonlinear adaptive evolutionary systems", *Quantitative Finance*, 1, 149–167.

A6: Gaunersdorfer, A. (2000), "Endogenous fluctuations in a simple asset pricing model with heterogeneous agents", *Journal of Economic Dynamics and Control*, 24, 799–831.

A7: Hommes, C.H., Huang, H. and Wang, D. (2005), "A robust route to randomness in a simple asset pricing model", *Journal of Economic Dynamics and Control*, 29, 1043–72.

L1: De Grauwe, P. and Grimaldi, M. (2004), "Bubbles and crashes in a behavioural finance model", Working Paper n. 7, DISEFIN, Università di Genova.

De Grauwe, P., Dieci, R. and Grimaldi, M. (2005), "Fundamental and non-fundamental equilibria in the foreign exchange market: A behavioural finance framework", CESifo Working Paper 1431.

L2: Beine, M., De Grauwe, P. and Grimaldi, M. (2006), "The impact of FX central bank intervention in a noise trading framework", CESifo Working Paper 1520.

L3: De Grauwe, P. and Grimaldi, M. (2005), "Heterogeneity of agents, transactions costs and the exchange rate", *Journal of Economic Dynamics and Control*, 29, 691–719.

L4: De Grauwe, P. and Grimaldi, M. (2006), "Exchange rate puzzles: A tale of switching attractors", *European Economic Review*, 50, 1–33. 691–719.

L5: Grimaldi, M. and Zwinkels, R.C.J. (2005), "Heterogeneous agents and non-normal fundamentals", Working Paper, Sveriges Riksbank and Radboud University Nijmegen.

S1: He, X.-Z. and Li, Y. (2007), "Power-law behaviour, heterogeneity, and trend chasing", *Journal of Economic Dynamics and Control*, 31, 3396–3426.

S2: He, X.-Z. (2003), "Asset pricing, volatility and market behaviour: A market fraction approach", Working Paper 95, Quantitative Finance Research Center, University of Technology, Sydney.

S3: Chiarella, C., Dieci, R. and Gardini, L. (2002), "Speculative behaviour and complex asset price dynamics: A global analysis", *Journal of Economic Behavior and Organization*, 49, 173–197.

S4: Dieci, R., Foroni, I., Gardini, L. and He, X.-Z. (2005), "Market mood, adaptive beliefs and asset price dynamics", Working Paper 162, Quantitative Finance Research Center, University of Technology Sydney.

S5: Chiarella, C., He, X.-Z. and Wang, D. (2006), "Statistical properties of a heterogeneous asset price model with time-varying second moment", in A. Namatame, T. Kaizouji and Y. Aruga (eds.), *The Complex Networks of Economic Interactions.* Berlin: Springer; pp. 109–123.

S6: Chiarella, C. and He, X.-Z. (2003), "Heterogeneous beliefs, risk, and learning in a simple asset-pricing model with a market maker", *Macroeconomic Dynamics*, 7, 503–536.

S7: Chiarella, C., He, X.-Z. and Hommes, C. (2006), "A dynamic analysis of moving average rules", *Journal of Economic Dynamics and Control*, 30, 1729-53.

5 References

Anderson, S., de Palma, A. and Thisse, J. (1993), *Discrete Choice Theory of Product Differentiation.* Cambridge, Mass.: MIT Press.

Arthur, W.B., Holland, J.H., LeBaron, B., Palmer, R. and Taylor, P. (1997), "Asset pricing under endogenous expectations in an artificial stock market", in W.B. Arthur, S.N. Durlauf and D.A. Lane (eds), *The Economy as an Evolving Complex System*, vol. II. Reading, MA: Addison-Wesley; pp. 15–44.

Beine, M., De Grauwe, P. and Grimaldi, M. (2006), "The impact of FX central bank intervention in a noise trading framework", CESifo Working Paper 1520.

Beja, A. and Goldman, M.B. (1980), "On the dynamic behavior of prices in disequilibrium", *Journal of Finance*, 35, 235–247.

BOSWIJK, GRIFFIOEN, P.G. AND HOMMES, C. (2000), "Success and failure of technical trading strategies in the cocoa futures market", Tinbergen Institute Discussion Paper, TI 2001–016/1

BROCK, W., LAKONISHOK, J. AND LEBARON, B. (1992), "Simple technical trading rules and the stochastic properties of stock returns", *Journal of Finance*, 47, 1731–64.

CHIARELLA, C. (1992), "The dynamics of speculative behaviour", *Annals of Operations Research*, 37, 101–123.

CHIARELLA, C., DIECI, R. AND GARDINI, L. (2002), "Speculative behaviour and complex asset price dynamics: A global analysis", *Journal of Economic Behavior and Organization*, 49, 173–197.

CHIARELLA, C. AND HE, X.-Z. (2001), "Asset price and wealth dynamics under heterogeneous expectations", Working Paper, School of Finance and Economics, University of Technology, Sydney.

CHIARELLA, C. AND HE, X.-Z. (2002), "Heterogeneous beliefs, risk, and learning in a simple asset-pricing model", *Computational Economics*, 19, 95–132.

CHIARELLA, C. AND HE, X.-Z. (2003), "Heterogeneous beliefs, risk, and learning in a simple asset-pricing model with a market maker", *Macroeconomic Dynamics*, 7, 503–536.

CHIARELLA, C., HE, X.-Z. AND HOMMES, C. (2006), "A dynamic analysis of moving average rules", *Journal of Economic Dynamics and Control*, 30, 1729-53.

CHIARELLA, C., HE, X.-Z. AND WANG, D. (2006a), "Statistical properties of a heterogeneous asset price model with time-varying second moment", in A. Namatame, T. Kaizouji and Y. Aruga (eds.), *The Complex Networks of Economic Interactions*. Berlin: Springer; pp. 109–123.

CHIARELLA, C., HE, X.-Z. AND WANG, D. (2006b), "A behavioural asset pricing model with a time-varying second moment", Working Paper 141, Quantitative Finance Research Center, University of Technology, Sydney.

CHIARELLA, C., HE, X., HUNG, H. AND ZHU, P. (2006), "An analysis of the cobweb model with boundedly rational heterogeneous producers", *Journal of Economic Behavior and Organization*, 61, 750–768.

Day, R.H. (1997), "Complex dynamics, market mediation and stock price behavior", *North American Actuarial Journal*, 1:3, 6–21.

De Grauwe, P., Dieci, R. and Grimaldi, M. (2005), "Fundamental and non-fundamental equilibria in the foreign exchange market: A behavioural finance framework", CESifo Working Paper 1431.

De Grauwe, P. and Grimaldi, M. (2004), "Bubbles and crashes in a behavioural finance model", Working Paper n. 7, DISEFIN, Università di Genova.

De Grauwe, P. and Grimaldi, M. (2005), "Heterogeneity of agents, transactions costs and the exchange rate", *Journal of Economic Dynamics and Control*, 29, 691–719.

De Grauwe, P. and Grimaldi, M. (2006), "Exchange rate puzzles: A tale of switching attractors", *European Economic Review*, 50, 1–33.

Dieci, R., Foroni, I., Gardini, L. and He, X.-Z. (2005), "Market mood, adaptive beliefs and asset price dynamics", Working Paper 162, Quantitative Finance Research Center, University of Technology Sydney.

Fleming, J., Kirby, C. and Ostdiek, B. (2001), "The economic value of volatility timing", *Journal of Finance*, 56, 329–252.

Farley, A. (2007), *Technical Analysis Tutorial—Moving Average Crossovers*; `http://www.tradingday.com/c/tatuto/movingaveragecrossovers.html`.

Farmer, J.D. (2001), "Toward agent-based models for investment", Association for Investment Management and Research, pp. 61–70; `http://www.aimr.org`.

Franke R. (1992), "Adaptive expectations and extrapolative forecasts: An approximate relationship", Working Paper, Department of Economics, University of Bielefeld.

Franke R. and Sethi R. (1993), "Cautious trend-seeking and complex asset price dynamics: Extended version", Working Paper, Department of Economics, University of Bielefeld.

Franke R. and Sethi R. (1998), "Cautious trend-seeking and complex asset price dynamics", *Research in Economics (Ricerche Economiche)*, 52, 61–79.

Gaunersdorfer, A. (2000a), "Endogenous fluctuations in a simple asset pricing model with heterogeneous agents", *Journal of Economic Dynamics and Control*, 24, 799–831.

Gaunersdorfer, A. (2000b), "Adaptive beliefs and the volatility of asset prices", *Central European Journal of Operations Research*, 29, 5–30.

Gaunersdorfer, A. and Hommes, C.H. (2007), "A nonlinear structural model for volatility clustering", In: G. Teyssière and A.P. Kirman (eds.), *Long Memory in Economics*. Berlin: Springer; pp. 265–288. (quoted from CeNDEF Working Paper 05–02, University of Amsterdam).

Gaunersdorfer, A., Hommes, C.H. and Wagener, F.O.O. (2003), "Nonlocal onset of instability in an asset pricing model with heterogeneous agents", CeNDEF Working Paper 03–10, University of Amsterdam.

Gaunersdorfer, A., Hommes, C.H. and Wagner, F.O.O. (2006), "Bifurcation routes to volatility clustering under evolutionary learning", *Journal of Economic Behavior and Organization* (forthcoming).

Grimaldi, M. and Zwinkels, R.C.J. (2005), "Heterogeneous agents and non-normal fundamentals", Working Paper, Sveriges Riksbank and Radboud University Nijmegen.

Gu, M. (1995), "Market mediating behavior: An economic analysis of the security exchange specialists", *Journal of Economic Behavior and Organization*, 27, 237–256.

He, X.-Z. (2003), "Asset pricing, volatility and market behaviour: A market fraction approach", Working Paper 95, Quantitative Finance Research Center, University of Technology, Sydney. [He_UTS95.pdf]

He, X.-Z. and Li, Y. (2007), "Power-law behaviour, heterogeneity, and trend chasing", *Journal of Economic Dynamics and Control*, 31, 3396–3426.

Hommes, C.H. (2001), "Financial markets as nonlinear adaptive evolutionary systems", *Quantitative Finance*, 1, 149–167.

Hommes, C.H., Huang, H. and Wang, D. (2005), "A robust route to randomness in a simple asset pricing model", *Journal of Economic Dynamics and Control*, 29, 1043–72.

Incademy.com (2007), "Technical analysis II. Classical interpretation of moving averages"; `http://www.incademy.com/courses/Technical-analysis-II/Average-share-prices/6`.

James, J. (2005), "FX trading models—how are they doing?", *Quantitative Finance*, 5, 425–431.

Kwon, K.-Y. and Kish, R.J. (2002), "Technical trading strategies and return predictability", *Applied Financial Economics*, 12, 639–653.

LeBaron, B., Arthur, W.B. and Palmer, R. (1999), "Time series properties of an artificial stock market", *Journal of Economic Dynamics and Control*, 23, 1487–1516.

LONGERSTAEY, J., ZANGARIAND, A. AND HOWARD, S. (1996), "RiskMetrics^{TM} — Technical Document", Morgan Guaranty Trust Company of New York: New York (4th ed.); downloadable (e.g.) from `www.riesgofinanciero.com/Riskmetrics_2.pdf`.

MANSKI, C. AND MCFADDEN, D. (1981), *Structural Analysis of Discrete Data With Econometric Applications.* Cambridge, Mass.: MIT Press.

TAYLOR, M.P. AND ALLEN, H. (1992), "The use of technical analysis in the foreign exchange market", *Journal of International Money and Finance*, 11, 304–314.

WESTERHOFF, F.H. (2004), "Market depth and price dynamics: A note", *International Journal of Modern Physics C*, 7, 1005–1012.