On the Interpretation of Price Adjustments and Demand in Asset Pricing Models with Mean-Variance Optimization

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Abstract

With reference to the class of asset pricing models with a market maker and mean-variance optimization of speculative agents, the note seeks to clarify the concepts behind the price adjustment rule, which are often treated somewhat carelessly in this literature. Calling attention to the distinction between the agents' desired holding of the risky asset and the desired change in their position, the following conclusion is drawn. If market prices are said to adjust in the direction of excess demand, then the story of the maximization of expected wealth should be dropped. On the other hand, the story could be perfectly maintained if the market maker were assumed to adjust prices inversely to his accumulated inventory.

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1. Introduction

This note is concerned with the interpretation of asset pricing models that, first, employ the figure of the market maker to adjust market prices and, second, whose speculative agents are said to be myopic mean-variance optimizers of expected wealth or, what amounts to the same, they maximize a CARA utility function of expected wealth.[1] These models are often not sufficiently clear about their precise notion of demand and the corresponding changes in the assets the agents are holding. The note makes a few elementary observations in this respect which, although in one formulation or another they are possibly common knowledge, are hardly ever made an explicit subject of discussion.

2. The solution to the mean-variance optimization

The presentation of the mean-variance optimization approach usually begins with the equation for the wealth dynamics. For concreteness, let the risky asset be a large stock or market index that pays a dividend y_t per share at the beginning of the market period t, and let the risk-free asset pay a fixed rate of return r. Then the evolution of the wealth W_h of the agents of type h from period t to period $t{+}1$ is described as

$$W_{h,t+1} \;=\; (1{+}r)\, W_{h,t} \;+\; [p_{t+1} + y_{t+1} - (1{+}r)p_t]\, z_{h,t} \tag{1}$$

where in addition p_t is the (uniform) price at which the asset is traded in period t, and $z_{h,t}$ is referred to as "the number of shares of the risky asset *purchased*" at time t (emphasis added). The latter quotation can be found in Hommes et al. (2005, p. 1046) or He and Li (2007, p. 3400), and there are several other papers speaking of "purchasing" or "buying".

More exactly, $z_{h,t}$ is a desired quantity which, however, can always be realized in these models. The optimization problem itself uses (1) as its wealth constraint, subject to which the expected value of a CARA utility function $U = U(W_{h,t+1}) = -\exp(-\alpha_h\, W_{h,t+1})$ is to be maximized, or in an equivalent formulation the term $E_{h,t}(W_{h,t+1}) - (\alpha_h/2)\, V_{h,t}(W_{h,t+1})$ (where $\alpha_h > 0$ is the agents' risk aversion coefficient and $E_{h,t}$, $V_{h,t}$ are their conditional expecta-

[1] CARA stands for constant absolute risk aversion.

tions and conditional variance). The explicit solution to this problem is

$$z_{h,t} \;=\; \frac{E_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]}{\alpha_h\, V_{h,t}[p_{t+1} + y_{t+1} - (1+r)p_t]} \tag{2}$$

If one goes back to the derivation of eq. (1), the formulation that $z_{h,t}$ is the number of shares *purchased* at time t is found to be somewhat careless. To see this, let, respectively, $A_{h,t}$ and $B_{h,t}$ be the number of shares and the risk-free asset (bonds, with a price of unity) which are in the portfolio of the agents of type h at the beginning of period t. At that time the agents have also received the dividends and interest payments, so their wealth is (dropping the index h)

$$W_t \;=\; p_t\, A_t \,+\, B_t \,+\, y_t\, A_t \,+\, r\, B_t \tag{3}$$

The agents invest their income on shares and bonds, while for speculative reasons they may also exchange shares for bonds or *vice versa*. Denote the shares and bonds they wish to hold at the beginning of the next period as A^d_{t+1} and B^d_{t+1}. The shares are bought at the current price p_t, taking into account the budget constraint

$$p_t\, A^d_{t+1} \,+\, B^d_{t+1} \;=\; W_t \tag{4}$$

After the new dividends and interest receipts are paid out and the market price of the shares has changed, the wealth at the beginning of $t{+}1$ amounts to (3) dated one period forward (and A^d, B^d in place of A and B). Using (4), rearrangement of this equation leads to

$$\begin{aligned} W_{t+1} &= p_{t+1}\, A^d_{t+1} \,+\, B^d_{t+1} \,+\, y_{t+1}\, A^d_{t+1} \,+\, r\, B^d_{t+1} \\ &= (1{+}r)(p_t\, A^d_{t+1} \,+\, B^d_{t+1}) \,+\, [p_{t+1} + y_{t+1} - (1{+}r)p_t]\, A^d_{t+1} \\ &= (1{+}r)\, W_t \,+\, [p_{t+1} + y_{t+1} - (1{+}r)p_t]\, A^d_{t+1} \end{aligned} \tag{5}$$

It follows that $z_{h,t}$ in (1) must be the desired *holding* of the risky asset. In contrast to the quotation in connection with (1), what the group of agents h can be reasonably supposed to "purchase" on the market is not the entire stock $A^d_{h,t+1}$ of the asset that they wish to hold, but just the difference from their actual holding $A_{h,t}$ at the beginning of period t.

3. Price changes in response to excess demand

The issue of the interpretation of $z_{h,t}$ in (2) is more serious than a careless use of words. To take up what has just been said, we consider it most natural to reserve the expression 'excess demand' for the aggregate differences between the agents' desired and actual holdings, and not for the sum of the desired holdings themselves. With H groups of speculative agents on the market, the period-t excess demand d_t is then given by

$$d_t \;=\; \sum_{h=1}^{H} \left(A^d_{h,t+1} - A_{h,t} \right) \tag{6}$$

Identifying $z_{h,t}$ with the desired holding $A^d_{h,t+1}$ of shares, it could now be argued that d_t is identical to $\sum_h z_{h,t}$ if the total number of shares, $\sum_h A_{h,t}$, remains constant and is conveniently set equal to zero. This is indeed a consistent situation in those mean-variance optimization models that employ a Walrasian auctioneer for continuous market clearing, $d_t \equiv 0$.[2] One has then only to be aware that in a deterministic equilibrium the income from dividends and interest is exclusively invested in bonds, which implies that the proportion of wealth held in shares is steadily decreasing. This is the price one has to pay for a CARA utility function in whose maximization the level of current wealth is eliminated.

Things are not that simple if market disequilibrium is admitted. A problem arises in the standard and small-scale models that abstain from rationing (in a batch auction or through an order book, say) but instead introduce a market maker, who has two tasks to fulfill. He absorbs any excess of supply from the market and serves any excess of demand from his inventory, and at the beginning of every period he quotes a new price. Regarding the latter task, the market maker is usually characterized as adjusting the price in the direction of the speculative traders' excess demand. With a positive adjustment coefficient μ_d, this reads,

$$p_{t+1} \;=\; p_t \;+\; \mu_d \, d_t \;=\; p_t \;+\; \mu_d \left(\sum_{h=1}^{H} A^d_{h,t+1} - \sum_{h=1}^{H} A_{h,t} \right) \tag{7}$$

[2] It might, however, happen that some agents hold short positions in terms of both the risky asset and bonds. If this were to be ruled out with certainty, a constraint would have to be introduced that requires the model to keep track of $A_{h,t}$ and even $B_{h,t}$; cf. Bottazzi et al. (2005, pp. 208f).

This formulation may be contrasted with the rule as it is written in the models with optimizing agents and a market maker (for example, the models in the two papers from which the quotation on the term $z_{h,t}$ in eq. (1) was taken):

$$p_{t+1} \;=\; p_t \;+\; \mu_d \sum_{h=1}^{H} z_{h,t} \tag{8}$$

Similar to the Walrasian framework, eq. (7) would be equivalent to (8) if $\sum_{h=1}^{H} A_{h,t}$ were to vanish. In the present context, however, it has to be taken into account that the assumption there employed, according to which the total number of shares is fixed, now also has to include the inventory $A_{m,t}$ of the market maker. In explicit terms, if $A_{h,t}$ and $A_{m,t}$ are conceived of as deviations from some target levels,

$$\sum_{h=1}^{H} A_{h,t} \;+\; A_{m,t} \;=\; 0 \tag{9}$$

Owing to the first task of the market maker, which is to buffer the aggregate excess demand, $A_{m,t}$ is obviously fluctuating over time in this identity. Therefore, in a market maker setting, the sum $\sum_{h=1}^{H} A_{h,t}$ in (7) *cannot* be treated as a constant. In other words, equations (7) and (8) are distinct and a claim that $z_{h,t}$ in (2) represents the agents' excess demand would be unwarrantable.

Equation (8) could be maintained as it is, with $z_{h,t}$ determined by (2), if this term is reinterpreted as the agents' desired *change* in their position. Clearly, (2) can then no longer be sold as the solution to the maximization of expected wealth. Nevertheless, a loss of the optimization flavour need not be too deeply regretted since the right-hand side of (2) makes perfect economic sense. It says, after all, that excess demand is proportional to the expected excess return on the asset and discounted by some measure of risk or volatility. Alternatively, (8) can be preserved if one finds another optimization problem that determines the desired change in the position and just yields (2) as its solution. In any case, the positions of the agents are here a mere appendix to the system, so there is no feedback that (in an otherwise stable stochastic setting) would prevent them from diverging.

Of course, speculative dynamics can also be built on the basis of eq. (7) with its position-based strategies $A^d_{h,t+1} = z_{h,t}$ from (2). This, however, leads to a type of model where $A^d_{h,t+1}$ as well as $A^d_{h,t} = A_{h,t}$ enter the price adjustment equation, which is more complicated and sets up a completely different framework from the one established by equations (2) and (8). In compensation for that, boundedness of the price guarantees boundedness of the agents' positions.[3]

4. An alternative price adjustment rule of the market maker

There is still another option to justify adherence to equations (2) and (8), namely, to drop eq. (7). This means the price adjustments must be supposed to follow a different principle from excess demand.

If one thinks of the accumulation of the market maker's inventory, it is only natural to consider the possibility that he will not remain completely passive in this respect. A minimal and straightforward idea has been proposed by Farmer (2001, p. 66), according to which a negative (positive) position that has been accumulated prompts the market maker to encourage selling (buying) by raising (lowering) the price more than usual. Let us here consider the case that fully concentrates on this new principle and has no more role for the excess demand to play. That is, replace eq. (7) with the rule,

$$p_{t+1} \;=\; p_t \;-\; \mu_a \, A_{m,t+1} \tag{10}$$

(μ_a a positive coefficient, of course). It is then easily checked that this equation gives rise to (8). In fact, since the change in the market maker's position $A_{m,t+1} - A_{m,t}$ is equal to $-d_t$, we have $p_{t+1} - p_t = -\mu_a \, A_{m,t+1} = -\mu_a \, (A_{m,t} - d_t) = -\mu_a \, (A_{m,t} - \sum_h A^d_{h,t+1} + \sum_h A_{h,t}) = -\mu_a \, (0 - \sum_h A^d_{h,t+1})$; the latter two equalities from (6) and (9). Hence eq. (10) becomes

$$p_{t+1} \;=\; p_t \;+\; \mu_a \sum_{h=1}^{H} A^d_{h,t+1} \tag{11}$$

which coincides with (8) if $A^d_{h,t+1}$ is identified with $z_{h,t}$. It follows that models working with the price adjustment rule (8) can maintain the conception of

[3] The approach with position-based (as opposed to order-based) strategies has been advanced by Doyne Farmer, although he is not interested in explicit optimization foundations for $A^d_{h,t+1}$; see, in particular, Farmer and Joshi (2002). The implications of the distinction between position-based and order-based strategies become clearer in Franke (2007), which puts forward a position-based prototype model at the same conceptual level as the influential order-based Beja–Goldman model.

$z_{h,t}$ as the solution to the mean-variance maximization of expected wealth if they include eq. (9) for the market maker's position in their framework and reinterpret (8) in terms of (10).

Incidentally, since all agents can realize their desired positions, $A_{h,t+1} = A^d_{h,t+1}$, it is now also seen that the excess demand rule (7) can be rewritten as

$$p_{t+1} \;=\; p_t \;-\; \mu_d\,(A_{m,t+1} - A_{m,t}) \tag{12}$$

Here it is not a negative position that prompts the market maker to encourage selling by raising the price, but a negative *change* in his position. From this point of view eq. (11) could be defended by arguing that the market maker, plausibly, gives priority to the *level effect* rather than to the *momentum effect* in (12). On the other hand, the stylized price adjustments in (7) may be said to be closer to more realistic trading protocols like a batch auction or an order book mechanism. Nevertheless, it is not this note's concern to stand up for one of these price adjustment rules but to ask for greater conceptual clarity in the literature employing them. Which of the two rules (7) or (10) is actually employed in a small-scale model may then be mainly a matter of convenience and analytical tractability.[4]

5. Conclusion

The present note discussed asset pricing models that combine a market maker and the speculative agents' mean-variance optimization of expected wealth, the focal point being that the authors are often somewhat careless in their reference to what they call 'demand' in the price adjustment rule. It was

[4]Indeed, once the distinction between the level and momentum effect in the market maker's price adjustments has been made, a combination of the two principles suggests itself. As remarked above, this option has been put forward by Farmer (2001). Franke (2007) adopts his rule in a setting with position-based strategies, and Franke and Asada (2008) in an otherwise Beja–Goldman setting where the speculative agents' strategies are oder-based (and the formulation of their market orders is complemented by a similar idea). Because of their prototype nature, the deterministic continuous-time versions of the latter two models are still mathematically tractable. Here we would like to put special emphasis on a complementarity result in the stochastic simulations in Franke (2007), which with $A^d_{h,t+1} = z_{h,t}$ is closest to the present framework: In a cyclical scenario it was found that neither of the two price adjustment principles should be given too much weight if the variability of positions as well as the price is to be kept within bounds.

clarified that the solution to the optimization problem is the desired *holding* of the asset, whereas the excess demand directed to the market maker will reasonably be the desired *change* in the asset. Although there is no need to respecify these models, for an appropriate interpretation of demand and price adjustments one is faced with the following alternative: either the present optimization story for the demand term is dropped or, maintaining it, the market maker is assumed to adjust prices (not proportionately to excess demand but) inversely to his accumulated inventory.

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